MODULE

8

Ratio and proportion 2

OBJECTIVES

This module is for study by an individual teacher or group of teachers. It:

- considers methods for solving problems involving ratio and proportion;
- discusses how mathematical problems and methods can be simplified or made more challenging to meet the needs of different pupils;
- highlights links between ratio and proportion and enlargement and similarity.

CONTENT

The module is in four parts.

1 Introduction

2 Solving problems on ratio and proportion

3 Enlargement and similarity

4 Summary

RESOURCES

Essential

- Your personal file for inserting resource sheets and making notes as you work through the activities in this module
- The *Framework for teaching mathematics: Years 7, 8 and 9*
- Video sequence 5, Year 6 pupils discussing test questions, from the DVD accompanying this module, and a DVD player
- A4 plain paper, scissors and glue, and a ruler
- A calculator
- The resource sheets at the end of this module:

  8a Key Stage 2 test questions involving ratio and proportion

  8b Year 6 pupils talking about test questions

  8c Different ways of solving a proportion problem

  8d Circle and diameter

  8e Metric paper sizes

  8f Summary and further action on Module 8

Desirable

- *Year 8 multiplicative relationships: mini-pack*
  http://www.standards.dfes.gov.uk/keystage3/respub/ma_intery8multi
- *Year 9 proportional reasoning: mini-pack*
  http://www.standards.dfes.gov.uk/keystage3/respub/ma_y9_prop

STUDY TIME

Allow approximately 90 minutes.

Part 1 Introduction

1 Modules 7 and 8 consider:

- how the ideas underlying ratio and proportion can be clarified;
- how links can be made between different mathematical topics that draw on ideas of ratio and proportion.

This module assumes that you have already studied Module 7.

2 The focus in Part 2 of this module is methods for solving problems on ratio and proportion. Usually the problem reduces to an equation of the form

$$a : b = c : d$$

where three of the numbers are known and the fourth has to be found. However, problems can appear in many different guises and several approaches are possible, some more sophisticated than others.

The focus for Part 3 of this module is enlargement.

As you consider approaches to ratio and proportion problems, there is a chance for you to reflect on:

- how the same problem might be approached by different pupils, depending on their level of understanding;
- how a problem can be simplified or made more challenging to suit the needs of different pupils;
- how pupils can move from informal to more formal methods.

Part 2 Solving problems on ratio and proportion

1 Look at **Resource 8a, Key Stage 2 test questions involving ratio and proportion**. The questions are from the May 2000 tests and pre-date the use of the euro in France.

- Test A (non-calculator) Question 15
- Test B (calculator) Question 21
- Test C (calculator) Question 1

Consider each question in turn. Think about and make a note of the possible alternative approaches and methods that could be used with each problem. Bear in mind whether or not the pupils will have had access to a calculator.

2 You now have an opportunity to watch responses from higher-attaining Year 6 pupils to the three questions you have just considered.

The pupils were interviewed in July, having completed the tests some weeks earlier in May 2000. Although the children are not a representative sample of pupils about to enter Key Stage 3, their responses illustrate issues that need to be addressed when teaching ratio and proportion in Years 7 to 9.

Get ready to watch **Video sequence 5, Year 6 pupils discussing test questions**.

Watch the video sequence for the first test question, then pause the video. Consider and make notes on the question on **Resource 8b, Year 6 pupils talking about test questions**. Then continue with the second and third questions similarly.

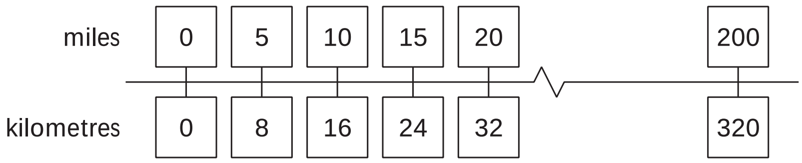
The whole video sequence lasts about 9 minutes: 3 minutes for the first question, 4 minutes for the second question and 2 minutes for the third question.

When you have finished watching, spend a few minutes completing the notes you have made on Resource 8b.

3 Read **Resource 8c, Different ways of solving a proportion problem**.

When pupils are working on ratio and proportion problems, teach them to do the following as a matter of routine.

- Consider whether the answer will be bigger or smaller than a given quantity.
  This is often sufficient to overcome common mistakes (e.g. in the Calais to Paris example, will the number of miles be more or less than 320?).
- Look at the numbers themselves.
  This will often help to identify the most efficient method of solution for a given problem.
- Set out their solution clearly.
  Often a picture or diagram can provide a powerful clue to the solution, as in the problem about the distance from Calais to Paris.



Notice how this diagram links to the use of the counting stick that Walt used in the lesson you watched in Module 7.

Whatever method they use, pupils need to understand and be able to express what they are doing at each stage of solving the problem.

Make sure that pupils think carefully about how to start a problem. Get them to compare different methods. Informal scaling or the four-cell method allows them to work within their understanding, provided that the numbers are amenable. Scaling, unitary and algebraic methods are more direct and work for any numbers.

Part 3 Enlargement and similarity

1 We will now consider the spatial aspect of proportion. This aspect of proportion is important; many pupils will develop their ability to reason through a proportional problem given in a spatial context.

Drawing out the links within mathematics allows pupils to develop a better understanding of concepts and moves away from the notion of isolated skills.

2 Consider one example of how links can be explored. Try the activities on **Resource 8d, Circle and diameter.**

3 Pupils who work on a visualisation task like the one on Resource 8d are likely to develop an *intuitive* awareness of the mathematical similarity of all circles and of the approximate relationship between circumference and diameter.

The visualisation task has an advantage over a practical, experimental approach. Measuring different circular objects, plotting a graph and estimating the relationship from the graph then requires pupils to observe a plausible pattern in experimental data. Here the result is apparent immediately. It can be related instantly to the circle and its circumference without the distraction of numbers and rounding.

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4 Now try the activity on **Resource 8e, Metric paper sizes**. You will need some sheets of A4 plain paper, scissors and glue, and a ruler.

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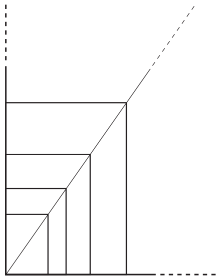
5 Compare your answers to the three problems on Resource 8e with the notes below.

Problem 1



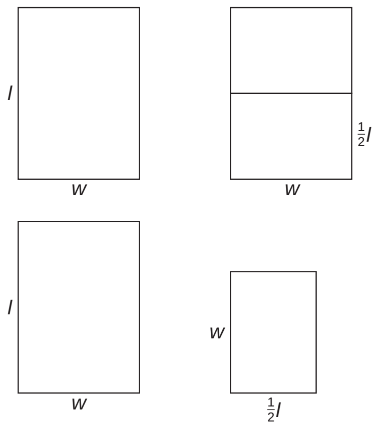
Problem 2

Let l = length of paper and w = width of paper.

$$\frac{l}{w} = \frac{w}{\frac{1}{2}l}$$

$$\tfrac{1}{2}l^2 = w^2$$

$$l^2 = 2w^2$$



Since the area scale factor is $\frac{1}{2}$, the scale factor for the lengths must be $\frac{1}{\sqrt{2}}$.

Problem 3

Using the results of the previous two problems, together with fact that A4 will be $\frac{1}{16}$ of the area of A0, the dimensions of A4 are found to be 210 mm by 297 mm.

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6 Like the work with strips of card in Module 7, diagrammatic work on enlargement of shapes provides a visual image for ratio and proportion. Diagrams can help pupils to understand scaling methods and the application of multipliers to solving problems (including inverse operations involving fractions, decimals and percentages).

It is worth noting the following points about the metric paper sizes problem.

- If pupils work on the problem of metric paper sizes, it is best if they start by investigating various rectangles of paper and confirm that folding them in half will not necessarily produce a mathematically similar shape. (Investigating old greetings cards may be the easiest way to do this.) This will help them to have a better appreciation of the uniqueness of the metric paper proportions.
- The metric paper proportion can be expressed in two ways:
  - as the ratio of length to width:
    $\frac{l_{A0}}{w_{A0}} = \frac{l_{A1}}{w_{A1}} = \dots$
  - as a scaling (enlargement) from one size to the next:
    $\frac{l_{A0}}{l_{A1}} = \frac{w_{A0}}{w_{A1}}$ and so on

If you get an opportunity to do so, observe the factors of enlargement shown on a photocopier to enlarge from one paper size to the next. These are usually given as percentages.

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7 Now study the supplement of examples, Framework section 4, pages 213–217. Look in particular at the Year 8 examples on enlargement and similarity, including the section on scale drawing.

To what extent do these examples correspond with your experiences as a teacher of mathematics? If there are differences, make a note of them in your points to discuss later with your head of department.

Think about how you could make the teaching of enlargement and similarity more explicit in your planning and teaching of Key Stage 3 lessons. Are there any examples in the Framework that would be useful to explore with the classes that you teach? Jot down your ideas in your personal file.

Part 4 Summary

1 Proportional reasoning is a key aspect of mathematics in Key Stage 3. It is important to:

- develop pupils' understanding of ratio, as distinct from fractions as part of a whole;
- ensure that pupils develop a range of effective methods for solving problems on proportion and are not reliant on just informal approaches;
- make explicit how proportional reasoning is used in different strands in mathematics and in other subjects, such as science, technology and geography: for example, there are direct links between enlargement and the graphical representation of functions and the measurement of gradients of lines, and also with trigonometry.

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2 Look back over the notes you have made during this module. Have you identified all the factors that you want to consider and adopt in your planning and teaching of ratio and proportion?

Use **Resource 8f, Summary and further action on Module 8**, to list key points you have learned, points to follow up in further study, modifications you will make to your planning or teaching, and any points to discuss with your head of department.

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3 If you have not already done so, you may find it useful to download and study the mini-packs on *Multiplicative relationships* and *Proportional reasoning*:

- *Year 8 multiplicative relationships: mini-pack*
  http://www.standards.dfes.gov.uk/keystage3/respub/ma_intery8multi
- *Year 9 proportional reasoning: mini-pack*
  http://www.standards.dfes.gov.uk/keystage3/respub/ma_y9_prop

Resource 8a Key Stage 2 test questions involving ratio and proportion

Consider each of these questions in turn. Think about and make a note of the possible alternative approaches and methods that could be made to each problem. Bear in mind whether or not the pupils will have had access to a calculator.

Test A Levels 3–5 Question 15 (calculator not allowed)

Peanuts cost 60p for 100 grams.
What is the cost of 350 grams of peanuts?

Possible approaches or methods to use for this problem:

Raisins cost 80p for 100 grams.
Jack pays £2 for a bag of raisins.
How many grams of raisins does he get?

Possible approaches or methods to use for this problem:

Test B Levels 3–5 Question 21 (calculator allowed)

A map shows that the distance from Calais to Paris is 320 kilometres.
5 miles is approximately 8 kilometres.
Use these facts to calculate the approximate distance in miles from Calais to Paris.

Possible approaches or methods to use for this problem:

Samira bought a present in France.
She paid 44.85 French francs for it.
9.75 French francs equal £1.
What was the cost of the present in pounds and pence?

Possible approaches or methods to use for this problem:

Test C Level 6 Question 1 (calculator allowed)

Shortcrust pastry is made using flour, margarine and lard.
The flour, margarine and lard are mixed in the ratio 8 : 3 : 2 by weight.

How many grams of margarine and lard are needed to mix with 200 grams of flour?

Possible approaches or methods to use for this problem:

Resource 8b Year 6 pupils talking about test questions

Watch the video sequence for each question separately. Then consider and make notes on the questions below.

1 Test A (non-calculator) Question 15

The sequence lasts 3 minutes.

How do the pupils' methods compare with those that you considered when you were studying the questions?

Where the pupils made errors, what would you do help them?

2 Test B (calculator) Question 21

The sequence lasts 4 minutes.

How do the pupils' methods compare with those that you considered when you were studying the questions?

Where the pupils made errors, what would you do help them?

3 Test C (calculator) Question 1

The sequence lasts 2 minutes.

How do the pupils' methods compare with those that you considered when you were studying the questions?

Where the pupils made errors, what would you do help them?

After watching the video

After you have finished watching the video, consider these questions.

How would the approaches you have seen in the video differ from what you would expect of pupils in Years 7, 8 and 9?

Are there any modifications that you want to make to your planning or teaching as a result of seeing and reflecting on the video extracts? If so, make some notes here on what you want to do.

Resource 8c Different ways of solving a proportion problem

There are several different ways of solving a proportion problem. The methods commonly used by pupils in the later stages of Key Stage 2 and in Key Stage 3 include:

- an informal scaling method;
- finding the scale factor;
- the unitary method;
- an algebraic method.

Informal scaling methods

Informal scaling methods are quite commonly seen, particularly as a mental or informal written approach. For example:

> *5 miles is about 8 km.*
> *So 10 miles is about 16 km.*
> *20 miles is about 32 km.*
> *200 miles is about 320 km.*

Such methods work well, provided that the scale factor is not difficult to find. Several of the Year 6 test examples in the video illustrate how the method can fail and the pupil is then forced to seek a more direct method.

Finding the scale factor

Finding the scale factor by division is more direct.

With the use of a calculator, this method is generally applicable whatever the numbers involved. However, pupils are not always able to express clearly and correctly what the answer to the division represents. For example:

> *320 km is 320 ÷ 8 = 40 times as far as 8 km.*
> *So 320 km is approximately 40 × 5 = 200 miles.*

For example, one pupil divided 320 by 8 to get 40. On being asked what the 40 represented, she said '40 kilometres'. Another was able to say 'how many lots of 8 kilometres there are in 320 kilometres', recognising that the 40 is a dimensionless number (a multiplier or scale factor). This is a problem about understanding ratio. It is distinct from the other sense of a fraction as part of a whole, as in a question such as: 'I have travelled one eighth of the journey from Calais to Paris. How far have I travelled?'. This question requires the same calculation, but the answer is clearly a distance.

The four-cell method

This method provides some structure to the informal scaling and scale factor approaches. It involves creating a four-cell diagram and labelling the columns with the two variables. Three of the four cells are filled in, and the fourth has to be found.

miles	km
5	8
?	320

$\downarrow \times 40$

Since 8 has to be multiplied by 40 to make 320, the 5 must also be multiplied or 'scaled' by the factor 40, to give 200. The answer is therefore 200 miles.

A four-cell diagram provides a useful starting point for organising the information in a structured way to make the relationships between the numbers more obvious, and to identify the scale factor and calculation needed. The diagram can also act as a reminder that corresponding quantities in direct proportion problems need to be in the same units.

The unitary method

The unitary method is commonly taught in Key Stage 3. For example:

8 km is approximately 5 miles.
So 1 km is approximately $\frac{5}{8}$ *miles.*
So 320 km is approximately $\frac{5}{8} \times 320 = 200$ *miles.*

This method too is generally applicable. Like the scaling method, pupils often use it even when it has not been formally taught. It was not used by any pupils in the video, but this may well be because it involves calculating with fractions, rather than for any other reason. The unitary method involves working out a rate, which is the change in one quantity per unit of the other.

A potential stumbling block is at the first stage of the solution. In this example, it is necessary to see that it is the number of miles per kilometre that is required, not the number of kilometres per mile. Of course, in many problems the rate is given in the data, for example: 'How far will I cycle in $3\frac{1}{2}$ hours at an average speed of 12 mph?'

The algebraic method

For the algebraic method, there are several ways in which an equation can be formed. For example:

Let x = number of miles in 320 kilometres. Then

$$\frac{x}{320} = \frac{5}{8}$$

leading to $x = \frac{5}{8} \times 320$

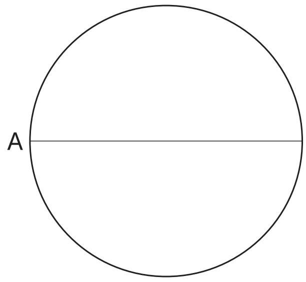
This is a more sophisticated method, appropriate to higher-attaining pupils, say in Year 9. It is necessary for more advanced work, for example in trigonometry.

Resource 8d Circle and diameter

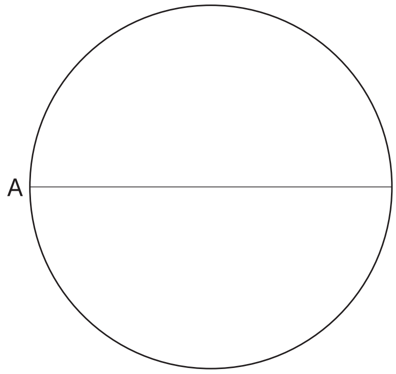
Look at the circle below.

Imagine a piece of string whose length is equal to the diameter, with one end fixed at A.

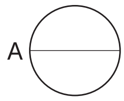
Imagine the string is wrapped in a clockwise direction around the circumference of the circle until it is taut. Mark the point that you think the other end of the piece of string will reach.



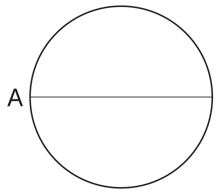
Repeat with this circle.



And this one.



And this one.



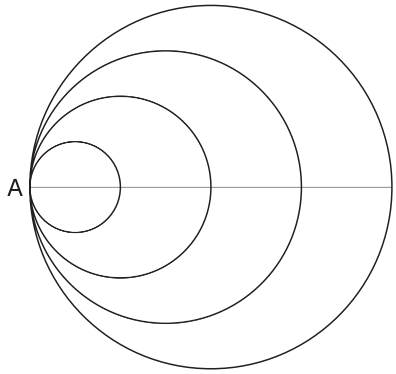
Do you agree that your marked point in each case is more than a quarter but less than half the circumference from A?

Let us say that the diameter is always roughly one third of the circumference, or:

$$\frac{C}{d} \approx 3$$

This is an example of a proportion: the circumference of a circle is proportional to its diameter.

Now repeat the activity on the series of nested circles below.



What do you notice about your set of marked points on the set of nested circles?

What do you think pupils would gain from a task like this?

Resource 8e Metric paper sizes

You will need some sheets of A4 plain paper, scissors and glue, and a ruler.

1 Start with two A4 sheets of paper. By successive folding and cutting one of the sheets, produce smaller paper sizes – A5, A6, and (although not commercially recognised) A7, A8, A9, A10.

Put the seven paper sizes in order and align them, with their longer sides in the same direction, so that the sheets meet at a corner. Take this corner point as a centre of enlargement. Draw radiating lines from this corner as a visual check that the sheets represent successive enlargements.

Construct a table of measurements – length, width and length/width.

Confirm that the scale factor of enlargement is approximately 1.4.

2 A sheet of metric paper of size A0 has an area of 1 m^2. The next smaller size, A1, is obtained by halving an A0 sheet, by folding along a line parallel to the shorter side. Subsequent sizes, A2, A3, A4, A5 and A6, are produced in a similar way.

The dimensions of the A0 sheet are such that A1, A2, …, have their sides in the same proportion.

Find the ratio of length to width of a sheet of metric paper.

3 Calculate the length and width of a sheet of A4 paper, correct to the nearest millimetre.

Resource 8f Summary and further action on Module 8

Look back over the notes you have made during this module. Identify the most important things to consider and modify in your planning and teaching of ratio, proportion and proportional reasoning.

List two or three key points that you have learned.

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List two or three points to follow up in further study.

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List two or three modifications that you will make to your planning or teaching of ratio, proportion and proportional reasoning.

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List the most important points that you want to discuss with your head of department, or any further actions you will take as a result of completing this module.

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