MODULE

4

Making links in algebra

OBJECTIVES

This module is for study by an individual teacher or group of teachers. It:

- considers the development of work on sequences, functions and graphs in Key Stage 3;
- considers links across different aspects of algebra and links with other strands of the mathematics curriculum;
- discusses the use of challenging activities to develop pupils' algebraic reasoning and use of algebra in solving problems.

CONTENT

The module is in five parts.

1 Using algebra to solve problems

2 Generalising

3 Linking sequences, functions and graphs

4 Looking at a lesson

5 Summary

RESOURCES

Essential

- Your personal file for inserting resource sheets and making notes as you work through the activities in this module
- The *Framework for teaching mathematics: Years 7, 8 and 9*
- Video sequence 2, a Year 8 algebra lesson, from the CD-ROM accompanying this module
- The resource sheets at the end of this module:

  4a Sequences, functions and graphs: definitions and examples

  4b Fibonacci chains

  4c Generalising

  4d Squares in a cross

  4e Graphs of linear functions

  4f Julie's lesson

  4g Summary and further action on Module 4

Desirable

- *Interacting with mathematics in Key Stage 3: Constructing and solving linear equations*
  http://www.standards.dfes.gov.uk/keystage3/respub/ma_interlin
- *Teaching and learning algebra pre-19*, a joint report from the Royal Society and the Joint Mathematical Council
  http://www.royalsoc.ac.uk/document.asp?id=1910

STUDY TIME

Allow approximately 90 minutes.

 DfES 0156-2004 G |

Part 1 Using algebra to solve problems

1 Algebra in Years 7 to 9 includes equations, formulae and identities, and sequences, functions and graphs. Module 3 focuses on some aspects of equations, formulae and identities. Module 4 focuses on sequences, functions and graphs. During this module, you will watch a Year 8 lesson linking algebra and geometry.

2 You are probably familiar with the definitions of a sequence, a function and a graph. By the end of Key Stage 3, pupils need to be aware of and understand these definitions. Check your understanding against the definitions on **Resource 4a, Sequences, functions and graphs: definitions and examples**.

3 Pupils have been developing their ideas about pattern in number throughout Key Stages 1 and 2. One aspect of this work relates to number properties and sequences, such as:

- one more than a multiple of 3;
- figurate numbers such as square numbers and triangular numbers;
- Fibonacci numbers.

In the problems on **Resource 4b, Fibonacci chains**, the number sequences have similar properties to the Fibonacci sequence – that is, each term is the sum of the previous two terms. Solve these problems and make some notes on a method that can be used to solve them.

4 While the answer to the first problem on Resource 4b is easy to spot, you probably used algebraic methods to solve the other problems.

All the sequences can be generalised in this form:

$a,\quad b,\quad a+b,\quad a+2b,\quad 2a+3b,\quad 3a+5b,\quad 5a+8b,\quad \ldots,$

where a is the first term and b is the second term. This allows the following equations to be formulated and solved:

$6 + 3b = 18$	3	4	7	11	18		
$5.66 + 3b = 25.91$	2.83	6.75	9.58	16.33	25.91		
$12 + 5b = 36$	4	4.8	8.8	13.6	22.4	36	
$30 + 8b = 4$	6	$^{-}3.25$	2.75	$^{-}0.5$	2.25	1.75	4

The four examples show that changing the first and last terms in sequences of this type alters the level in difficulty, so that an activity for pupils that is based on this task can readily be differentiated. For example, the first problem would be suitable for pupils working confidently at level 4, whereas the other three problems are more suitable for pupils working at level 5 or level 6.

5 Now study the supplement of examples, Framework section 4:

- pages 144–159, which focus on sequences;
- pages 6–9, which focus on problems involving number and algebra.

As you study the examples, identify more opportunities for using algebra to solve problems in Key Stage 3 mathematics lessons. Make a note of these examples in your personal file.

Part 2 Generalising

1 In the Fibonacci sequence problems in Part 1, it was possible to express each term of the sequences in a general form by using letters to stand for numbers.

One way of introducing pupils to algebraic generalisation is to ask them to extend number patterns. Answer the questions on **Resource 4c, Generalising**, which are typical of the problems that can be given to pupils working at level 5.

---

2 In the first problem on Resource 4c, you have probably described the nth line in the pattern by describing it as:

$(n - 1)(n + 1) = n^2 - 1$

or as:

$n(n + 2) = (n + 1)^2 - 1$

The pattern can be extended backwards to explore multiplication of negative numbers, since any integer, positive or negative, can be substituted for n.

$$\begin{aligned}
1 \times 3 &= 2^2 - 1 \\
0 \times 2 &= 1^2 - 1 \\
(^-1) \times 1 &= 0^2 - 1 \\
(^-2) \times 0 &= (^-1)^2 - 1 \\
(^-3) \times (^-1) &= (^-2)^2 - 1
\end{aligned}$$

What happens if fraction or decimal values are substituted for n? Is it still the case that $(n - 1)(n + 1) = n^2 - 1$?

An equation like $n^2 - 1 = (n - 1)(n + 1)$ that holds true for all possible values of the variables is called an identity.

The second problem on Resource 4c has a connection with the first problem, in that each of the four numbers 899, 3599, 10 403, 359 999 is 1 less than a perfect square. It can therefore be expressed in the form $n^2 - 1$, which factorises as $(n - 1)(n + 1)$. This helps to find the solutions:

$899 = 30^2 - 1 = 29 \times 31$
$3599 = 60^2 - 1 = 59 \times 61$
$10\,403 = 102^2 - 1 = 101 \times 103$
$359\,999 = 600^2 - 1 = 599 \times 601$

What happens with other values? Check that:

$(5.816)^2 - 1 = 4.816 \times 6.816$

and that:

$(^-15.216) \times (^-13.216) + 1 = (^-14.216)^2$

Giving pupils opportunities to explore generalities like these helps them to develop an understanding of the power of algebra.

Part 3 Linking sequences, functions and graphs

1 Connections between sequences, functions and graphs are often not given enough emphasis in Key Stage 3 mathematics lessons.

Pupils are often introduced to functions through 'number machines' or 'function machines'. In this first example, the rule is given, along with some input numbers. Pupils soon learn to work backwards intuitively from the output numbers.

in $\longrightarrow$ $\boxed{\text{multiply by 4 then add 1}}$ $\longrightarrow$ out

in	1	7	200	2.5			–2
out					121	2	

In Year 7 pupils are expected to begin to use algebraic notation:

$n \rightarrow 4n + 1$ or $y = 4x + 1$

The mapping can be separated out into single operation machines:

$x \longrightarrow \boxed{\times 4} \longrightarrow \boxed{+\,1} \longrightarrow y$

By Year 8, pupils should be able to transform this using inverse operations:

$x \longleftarrow \boxed{\div 4} \longleftarrow \boxed{-\,1} \longleftarrow y$

or: $x = \dfrac{y-1}{4}$

This links closely to the second form of machine, which is of the 'What went in?' type.

in $\longrightarrow$ $\boxed{\text{multiply by 2 then add 3}}$ $\longrightarrow$ out

in							
out	7	25	3	55	863	3.1	2

The function for this machine is $y = 2x + 3$ and its inverse is $x = \frac{1}{2}(y - 3)$.

Both types of machine, once introduced, would provide useful number practice in an oral and mental session. The examples show how operations on different numbers can be targeted for practice, and so questions can be matched to the stage of development of individual pupils.

A third type of number machine has input and corresponding output numbers and the function has to be found.

in $\longrightarrow$ $\boxed{}$ $\longrightarrow$ out

in	3	4	6	10	1.7		–5
out	9	13	21	37		0	

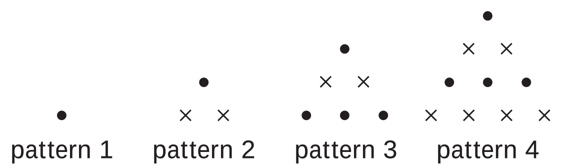
In this machine, the function is $y = 4x - 3$ and $x = \frac{1}{4}(y + 3)$.

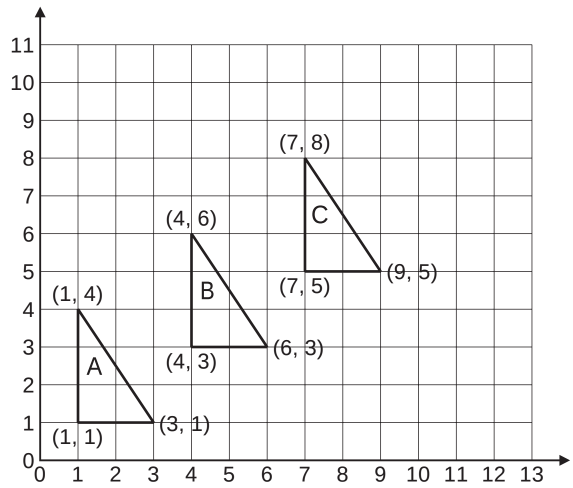
What number is unchanged by this machine? Or, to put this question another way, for what number are the input and output numbers the same? What are the 'stay the same' numbers for the first two types of number machines?

A possible investigation for pupils is to explore 'stay the same' numbers for different number machines.

---

2 Another context in which pupils will meet mappings such as these is in work on number sequences. Work on number sequences will have started in Key Stage 2. The two questions below are level 4 questions from the Key Stage 2 National Curriculum tests.

> This series of patterns grows in a regular way.
>
> 
>
>
> How many **dots** would be in pattern 5?
>
> How many **crosses** would be in pattern 5?

> 
>
>
> Write the coordinates of the next triangle in the sequence.

Look at the number sequence on **Resource 4d, Squares in a cross**, and answer the accompanying questions.

---

3 The function related to *Squares in a cross* is represented by the equation $y = 4x + 1$.

The large numbers in the table are beyond pupils' capacity to draw and count. This encourages them to move from particular cases to the general. Key Stage 3 pupils should first be asked to find the rule in words, to provide a step towards the algebra, and then to suggest a way of checking on its accuracy.

Two levels of generalisation emerge from these types of spatial patterns. For pupils, it is easier to spot links between successive terms, for example 'It goes up in fours', than to relate a term to its position in the sequence.

It helps pupils if you encourage them to justify and explain why a rule or relationship works in the context of the situation, relating back to the diagrams and not just to the pattern of numbers. One way of doing this is to ask them to calculate particular terms in the sequence. For example, in *Squares in a cross*, the 10th cross needs $10 \times 4 + 1$, or 41 squares, and the 100th cross needs $100 \times 4 + 1 = 401$ squares. From the particular examples, they are able to see that, in general, $y = 4x + 1$.

---

4 With number sequences based on spatial patterns like these, the values of the variables are whole-number values only. In a true algebraic relationship, the variables can take any values on a continuous scale. Graphs of number patterns should really be a set of separate points, but in order to look at the algebraic relationship we usually join the points as though they represent a continuous function.

Look at **Resource 4e, Graphs of linear functions**. One of the graphs represents the function $y = 4x + 1$. Identify which graph it is and then find the equations of the other graphs.

---

5 Pupils in Year 7 are expected to draw graphs such as those on Resource 4e. In Year 8, pupils are introduced to ideas of gradient and intercept.

Compare your answers to the questions on Resource 4e with those below.

A $x + y = 4$
B $y = 4x + 1$
C $y = 4x - 3$
D $y = \frac{1}{2}x + 3$
E $y = 2x + 3$

The graph for which the point of intersection with each line would be the 'stay the same' number for that function is the line $y = x$.

Linking elements of algebra together is part of the algebraic reasoning that needs to be developed throughout Key Stage 3. Pupils need to gain insight into the power and purpose of algebra as well as learning algebraic techniques.

---

6 Study the supplement of examples, Framework section 4, pages 160–177, which focus on functions and graphs.

As you study the examples, identify more opportunities for linking work on sequences, functions and graphs in Key Stage 3 mathematics lessons. Note these examples in your personal file.

Part 4 Looking at a lesson

1 Algebraic ideas and reasoning can support pupils' understanding of other areas of mathematics. There are, for example, strong links between algebra and geometric reasoning.

Get ready to watch **Video sequence 2, a Year 8 algebra lesson**, which exemplifies these links. The teacher is Julie.

You will notice that although Julie makes it clear early on that the letters represent the areas of the shapes (and not 'the shapes'), the pupils are not always so precise in their language. It is important that pupils understand that the letters represent values which, although unknown, can be handled like known numbers.

As you watch the lesson, focus particularly on the questioning styles adopted by Julie in her direct teaching. Use **Resource 4f, Julie's lesson**, to make notes.

The video sequence lasts about 11 minutes.

---

2 When you have finished watching, spend a few minutes completing the notes you have made on **Resource 4f**. Then think about how Julie's approach compares with what you would have done.

Part 5 Summary

1 Understanding the links between sequences, functions and graphs is a cornerstone of Key Stage 3 mathematics. Teaching has to help pupils to appreciate that algebra allows them to represent and explore general relationships and that this is more powerful than looking only at specific cases.

Pupils need opportunities to use their algebraic skills in problem solving in order to:

- increase their awareness of when and how algebra can be useful;
- improve their knowledge of algebraic conventions;
- deepen their understanding of algebraic rules;
- practise their use of algebraic techniques.

Most importantly, they need opportunities to see how algebra can provide insights into the underlying situation that the algebra is modelling.

---

2 Look back over the notes you have made during this module. Have you identified what you may need to consider and adopt in your planning and teaching of algebra?

Use **Resource 4g, Summary and further action on Module 4**, to list key points you have learned, points to follow up in further study, modifications you will make to your planning or teaching, and points to discuss with your head of department.

---

3 As mentioned in Module 3, if you are interested in reading more about the teaching of algebra in secondary schools, download *Teaching and learning algebra pre-19*, a joint report from the Royal Society and the Joint Mathematical Council, from http://www.royalsoc.ac.uk/document.asp?id=1910.

If you have not already done so, you could also download and look at the Key Stage 3 Strategy's *Interacting with mathematics in Key Stage 3: Constructing and solving linear equations,* from http://www.standards.dfes.gov.uk/keystage3/respub/ma_interlin.

Resource 4a Sequences, functions and graphs: definitions and examples

Sequences

A **sequence** is an ordered succession of terms formed according to a rule. There can be a finite or infinite number of terms.

The sequences most commonly considered in Key Stage 3 mathematics have:

- an identifiable mathematical relationship between the value of a term and its position in the sequence; and/or
- an identifiable mathematical rule for generating the next term in the sequence from one or more existing terms.

Examples
The squares of the integers: 1, 4, 9, 16, 25, …
The Fibonacci sequence: 1, 1, 2, 3, 5, 8, …

Functions

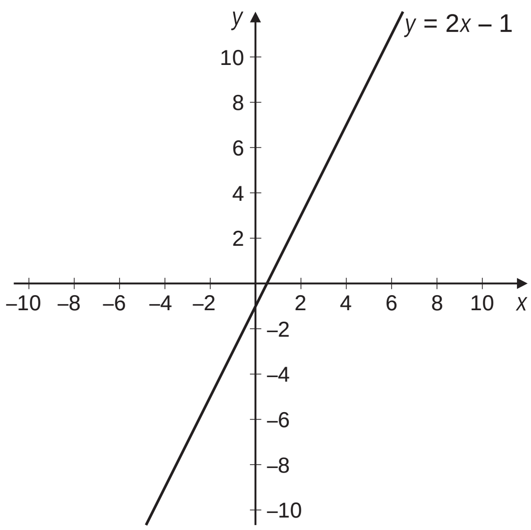
A **function** is a rule that associates each term of one set of numbers with a single term in a second set. The relationship can be written in different ways.

Example
$x \rightarrow 2x - 1$
$y = 2x - 1$

Graphs

A **graph** of a function is a diagram that represents the relationship between two variables or sets of numbers.

Example



Resource 4b Fibonacci chains

All these number chains have similar properties to the Fibonacci sequence – that is, each term is the sum of the previous two.

Find the missing terms.

3	…	…	…	18

2.83	…	…	…	25.91

4	…	…	…	…	36

6	…	…	…	…	…	4

Use this space for any working that you want to do.

Explain how to solve problems like these.

Resource 4c Generalising

1 Consider this pattern:

$1 \times 3 = 2^2 - 1$

$2 \times 4 = 3^2 - 1$

$3 \times 5 = 4^2 - 1$

$4 \times 6 = 5^2 - 1$

a. What will the next two lines be?

b. What will the 10th line be?

c. What will the 100th line be?

d. If I wanted to know what a particular row will be, say the *n*th row, how could you tell me?

2 Find a pair of factors of:

a. 899

b. 3599

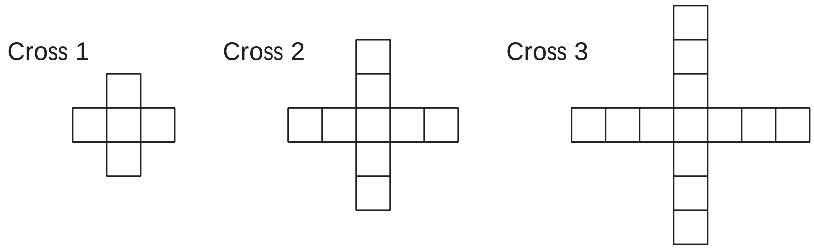
c. 10 403

d. 359 999

Make up two similar questions.

Resource 4d Squares in a cross

Fill in the missing values in the table by studying the patterns.



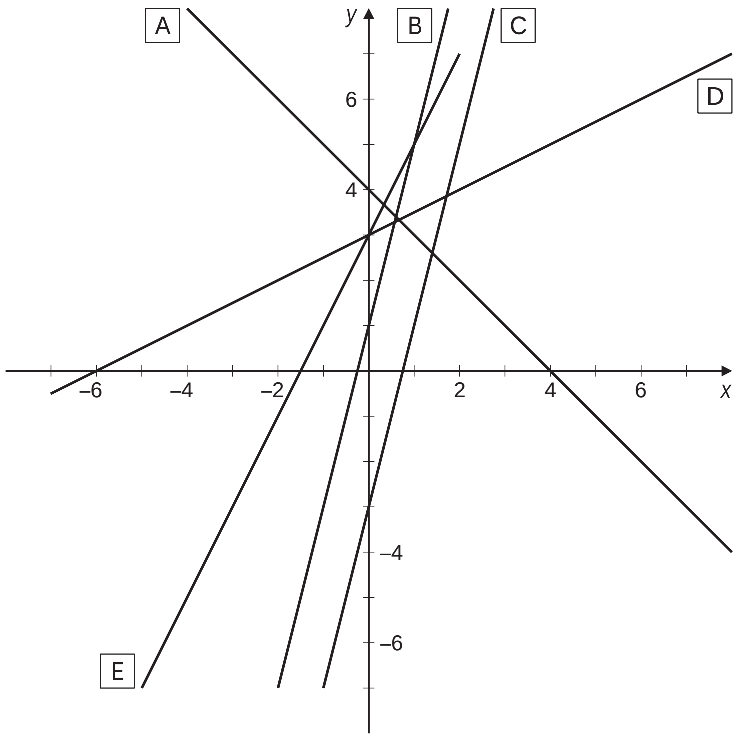
Number of cross	x	1	2	3	4	5	10	50		200
Number of squares	y								1001	

What is the rule which links the number of squares in any cross (y) to its position in the sequence (x)?

Explain why this is the rule by referring to the properties of the shapes.

Resource 4e Graphs of linear functions

Which line represents the function $y = 4x + 1$?



Find the equations of the other lines.

Line	Equation
A	
B	
C	
D	
E	

What other graph would you need to draw so that its point of intersection with each of the other lines would be the 'stay the same' number for that line's function?

Resource 4f Julie's lesson

How do Julie's questions help pupils to visualise and explain their solutions in different ways?

How does the activity help Julie's pupils to make links between algebra and geometry?

How valuable was the final plenary of the lesson in establishing those links?

Resource 4g Summary and further action on Module 4

Look back over the notes you have made during this module. Identify the most important things to consider and modify in your planning and teaching of algebra.

List two or three key points that you have learned.

-
-
-

List two or three points to follow up in further study.

-
-
-

List two or three modifications that you will make to your planning or teaching of algebra.

-
-
-

List the most important points that you want to discuss with your head of department, or any further actions you will take as a result of completing this module.

-
-
-

