

Thinking about algebra

OBJECTIVES

This module is for study by an individual teacher or group of teachers. It:

- considers some stimulating activities for teaching an aspect of algebra, the simplification of algebraic expressions;
- discusses how the activities can also help to develop algebraic reasoning;
- considers how the activities might be incorporated in mathematics lessons.

CONTENT

The module is in five parts.

- 1 Algebra in Key Stage 3
- 2 Practising collecting like terms
- 3 Applying algebraic reasoning
- 4 Approaches to algebra in the classroom
- 5 Summary

RESOURCES

Essential

- Your personal file for inserting resource sheets and making notes as you work through the activities in this module
- The *Framework for teaching mathematics: Years 7, 8 and 9*
- Scissors
- The resource sheets at the end of this module:
 - 3a An algebra loop card game
 - 3b An algebraic magic square
 - 3c Addition squares
 - 3d Reflections on addition squares
 - 3e More addition squares
 - 3f Approaches to algebra in the classroom
 - 3g Summary and further action on Module 3

Desirable

- *Framework for teaching mathematics from Reception to Year 6* (the Primary Framework), pages 9 and 10 of section 1, the introduction (pages 9 and 10, Laying the foundations for algebra, can be downloaded from http://www.standards.dfes.gov.uk/primary/publications/mathematics/math_framework/introduction/938949)
- *Interacting with mathematics in Key Stage 3: Constructing and solving linear equations* http://www.standards.dfes.gov.uk/keystage3/respub/ma_interlin
- *Teaching and learning algebra pre-19*, a joint report from the Royal Society and the Joint Mathematical Council www.royalsoc.ac.uk/document.asp?id=1910

STUDY TIME

Allow approximately 90 minutes.

Part 1 Algebra in Key Stage 3

- 1 This module considers one of the learning objectives from the Year 8 teaching programme for algebra: that pupils should be taught to simplify or transform linear expressions by collecting like terms. The module describes some activities that address this learning objective. It then goes on to consider the place of these and similar activities in mathematics lessons.

- 2 Algebra in Years 7 to 9 includes equations, formulae and identities, and sequences, functions and graphs. Read pages 14–15 of the Guide to the Framework, section 1, of the *Framework for teaching mathematics: Years 7, 8 and 9*. These pages summarise the Key Stage 3 Strategy's approach to the teaching of algebra.

The Framework makes clear that algebra in Key Stage 3 involves:

- developing pupils' understanding that algebra is a way of generalising either from arithmetic, or from particular cases or from patterns and sequences;
- providing regular opportunities to construct algebraic expressions and formulae and to transform one expression into another – for example, by collecting like terms, taking out common factors, working with inverses or solving linear equations;
- using opportunities to:
 - represent a problem and its solution in tabular, graphical or symbolic form, using a graphical calculator or a spreadsheet where appropriate;
 - relate solutions to the context of the problem;
- developing algebraic reasoning, including an appreciation that while a number pattern may suggest a general result, a proof is derived from the structure of the situation being considered.

- 3 Algebra is not taught formally in Key Stage 2. If you have access to it or can download the relevant pages, read pages 9 and 10 of the Introduction, section 1 in the Primary Framework. These pages summarise how the foundations for algebra are laid in Key Stage 2 through a variety of relevant experiences, although the word 'algebra' is not used to describe them.

Part 2 Practising collecting like terms

- 1 Spend a few minutes jotting down in your personal file the activities and contexts that you currently use to teach pupils to simplify or transform linear expressions by collecting like terms.

- 2 Copy, cut out and shuffle up the cards on **Resource 3a, An algebra loop card game**. Spread out the cards on a flat surface so that you can see them all. Take one of the cards at random and place it to the left of you. Find the answer to the question on this card and place it to the right of the first card to form a line. Carry on until you have used all 18 cards.

This is a self-checking activity in that the last card on the right of the line should link back to the first card on the left.

The activity is one that small groups of pupils can work on collaboratively. Alternatively, a similar set of cards can be distributed around a whole class to play an 'I have ... What is ...' loop card or 'follow me' game. A pupil starts the game by reading their algebraic

expression and their question. Other pupils follow until all cards have been called and the 'loop' has been completed. As pupils play the game, you can write each new expression on the board for everyone to see.

Take a few moments to think about these questions.

- What is the purpose of this kind of activity?
- What could the advantages be of using an activity like this at the start of a lesson?
- How could the activity be organised or adapted it to make it suitable for pupils at different levels of attainment?

Part 3 Applying algebraic reasoning

- 1** The activity in Part 2 of this module provides mental practice in collecting like terms. Every pupil is involved as they think about each question to see whether the answer matches the expression on their card. The complexity of the expressions can be varied and, where pupils play the game in groups, different sets of cards can be given to different groups.

Other activities offer further mental practice but also require reasoning to solve algebraic problems. An example of this kind of activity is one involving magic squares. In a magic square, a set of numbers is arranged in a square grid so that the numbers in each row, column and the two main diagonals have the same total.

In your personal file, quickly arrange the numbers 1 to 9 to form a 3 by 3 magic square. This problem helps pupils to practise adding three single-digit numbers mentally but, in addition, requires some reasoning in order to arrive at a solution.

A teacher who wants pupils who have solved this problem to describe their methods and reasoning is likely to prompt them with questions such as:

- Where did you start?
- What did you do next? Why?
- How many different solutions are there?

Through discussion, the teacher will help pupils to appreciate that although solutions may look different, they are all reflections and rotations of each other. There is only one solution represented in different ways. The central number is the middle number of the set of numbers 1 to 9 (and is also the mean of the set).

- 2** The same conclusion can be reached by algebraic reasoning.

Copy, cut out and shuffle up the nine cards, each with an algebraic expression, on **Resource 3b, An algebraic magic square**. Arrange the cards to form a magic square.

When you have completed the square to your satisfaction, consider these questions.

- How did you start? How did you continue?
- Could you have started and continued in a different way?
- How did you check your solution?
- How many different solutions can you find?
- To what extent did your earlier consideration of a magic square with the numbers 1 to 9 help you to solve the problem with algebraic expressions?

Some points to remember in relation to the algebraic magic square activity are these.

- The problem demands logical reasoning and other problem-solving skills.
- The solution can be checked by determining whether the sum of the three algebraic expressions in each line is the same. It can also be checked by substituting particular values for a and b (for example, $a = 1$, $b = 0$, or $a = 0$, $b = 1$).
- There is one solution. Reflections and rotations produce other possible arrangements of this solution.

$a + 4b$	$8a + 3b$	$3a + 8b$
$6a + 9b$	$4a + 5b$	$2a + b$
$5a + 2b$	$7b$	$7a + 6b$

- Substituting different values for a and b produces an infinite number of different magic squares, all with the same structure, rules and internal relationships as each other.
- This approach is one way of illustrating how the handling of algebraic expressions can grow out of familiarity with handling numbers. Working first with the original 1 to 9 number square helps to provide insights into solving the algebraic square.

- 3** The next activity involves collecting like terms in the context of addition grids. Once again, reasoning is involved.

Complete the addition squares and answer the questions on **Resource 3c, Addition squares**.

- 4** Compare your explanations and justifications on Resource 3c with those below.

- 1** *An explanation and justification of the solution to the second of the first pair of problems might go like this.*

For the top left entry:

$$97 + \square = 126$$

The number in the box must be 29, since $126 - 97 = 29$.

For the bottom left entry:

$$29 + \square = 178$$

*This time, the number in the box must be 149, since $178 - 29 = 149$...
... and so on.*

- 2** *An explanation and justification of the solution to the second of the second pair of problems might go like this.*

For the top left entry:

$$(3f + 4g) + (\square + \square) = 4f - 3g$$

The terms in the boxes must be f and $-7g$, giving the expression $f + -7g$ or $f - 7g$.

For the bottom left entry:

$$(f - 7g) + (\square + \square) = 3f - 5g$$

*This time, the terms in the boxes must be $2f$ and $2g$, giving the expression $2f + 2g$...
... and so on.*

Now consider the questions on **Resource 3d, Reflections on addition squares**.

5 Compare your answers to the questions on Resource 3d with those below.

- What do you notice about the diagonals in the addition squares on Resource 3c?
Each diagonal has the same total.
- Prove that this must always be the case for a 2 by 2 addition square.
A proof that each diagonal in a 2 by 2 addition square must have the same total can be derived from a square like this.

+	a	b
c	$a + c$	$b + c$
d	$a + d$	$b + d$

The total of the cells on the diagonal from top left to bottom right is
 $(a + c) + (b + d) = a + b + c + d$

The total of the cells on the diagonal from top right to bottom left is
 $(b + c) + (a + d) = a + b + c + d$

- What are the advantages of moving explicitly from numbers to algebra in activities like these?
Pupils can draw the parallels between the arithmetic and algebraic processes. Their understanding of what happens with numbers helps them to understand and generalise what happens with algebraic expressions.

6 Complete the addition squares and answer the questions on **Resource 3e, More addition squares**. This time you are given the 'output' expressions and need to think about what the missing 'input' expressions might be.

7 Compare your answers to the questions on Resource 3e with those below.

- How many solutions are there to each of the addition square puzzles?
All except the third example have an infinite number of solutions. The third example has no solution.
- Prove your statements about the number of solutions.
Proofs about the number of solutions might go like this.

The third example has no solution, since in a complete 2 by 2 addition square the diagonals must have the same total. In this example, each diagonal has a different total.

In the first, second and fourth puzzles, there is a limitless choice for the first and, therefore, subsequent inputs. For example, put an unrelated term (say z) as an input that can take any value. This produces a workable solution.

+	$3a + 5b - z$	$5a + 3b - z$
z	$3a + 5b$	$5a + 3b$
$z - a + 2b$	$2a + 7b$	$4a + 5b$

Part 4 Approaches to algebra in the classroom

- 1 Study the yearly teaching programmes for algebra, in Framework section 2.

As you do so, identify where objectives linked to the simplification or transformation of algebraic expressions occur.

- 2 Now study the supplement of examples, Framework section 4:

- pages 116–121, which focus on the learning objective ‘Simplify or transform algebraic expressions’.
- pages 30–35, which focus on using and applying mathematics.

As you study the examples, identify more opportunities for linking the rules of algebra to those of number in Key Stage 3 mathematics lessons. Make a note of these examples in your personal file.

- 3 Consider and make notes on the questions on **Resource 3f, Approaches to algebra in the classroom**.

Part 5 Summary

- 1 Some important principles in the teaching of algebra in Key Stage 3 are as follows.

- Key Stage 3 pupils need to develop understanding that algebra is a way of generalising.
It helps pupils when they are aware that what works with numbers works also with algebraic expressions.
- There are stimulating activities that can be developed from work with numbers and which allow Key Stage 3 pupils to practise simplifying or transforming algebraic expressions.
The Framework illustrates different examples that can be used in this way. Many of these activities also involve algebraic reasoning.
- Key Stage 3 pupils should be asked to explain their solutions to algebraic problems and to justify their mathematical reasoning.
Pupils working confidently at level 5 or above should be asked to prove their results.

- 2 Look back over the notes you have made during this module. Have you identified the most important things that you may need to consider and adopt in your planning and teaching of algebra?

Use **Resource 3g, Summary and further action on Module 3**, to list key points you have learned, points to follow up in further study, modifications you will make to your planning or teaching, and points to discuss with your head of department.

- 3 If you are interested in reading more about the teaching of algebra in secondary schools, download *Teaching and learning algebra pre-19*, a joint report from the Royal Society and the Joint Mathematical Council, from www.royalsoc.ac.uk/document.asp?id=1910. You may also find it useful to download and study *Interacting with mathematics in Key Stage 3: Constructing and solving linear equations* http://www.standards.dfes.gov.uk/keystage3/respub/ma_interlin.

Resource 3a An algebra loop card game

Copy, cut out and shuffle up the 18 cards below. Spread out the cards on a flat surface so that you can see them all.

Take one of the cards at random and place it to the left of you. Find the answer to the question on this card and place it to the right of the first card to form a line.

Carry on until you have used all 18 cards. The last card on the right of the line should link back to the first card on the left.

<p>I have $3a + 2b$</p> <p>What is $4b$ less?</p>	<p>I have $2a + 4b$</p> <p>What is $4a$ more?</p>	<p>I have $3a - 5b$</p> <p>What is $7b$ more than this?</p>
<p>I have $6a - 3b$</p> <p>What is one third of this?</p>	<p>I have $3a + 4b$</p> <p>What is $9b$ less than this?</p>	<p>I have $7a - 2b$</p> <p>What is $a + b$ less?</p>
<p>I have $3a + 5b$</p> <p>What is $a + b$ less than this?</p>	<p>I have $-a - b$</p> <p>What is $3b$ more?</p>	<p>I have $3a + 3b$</p> <p>What is $2b$ more than this?</p>
<p>I have $2b$</p> <p>What is $a + b$ more than this?</p>	<p>I have $2a - b$</p> <p>What is double this?</p>	<p>I have $2b - a$</p> <p>What is a more than this?</p>
<p>I have $6a + 4b$</p> <p>What is a more than this?</p>	<p>I have $7a + 4b$</p> <p>What is $4a$ less than this?</p>	<p>I have $a + 3b$</p> <p>What is $2a$ more than this?</p>
<p>I have $-2a - 2b$</p> <p>What is half of this?</p>	<p>I have $3a - 2b$</p> <p>What is $4a$ more?</p>	<p>I have $4a - 2b$</p> <p>What is $6a$ less than this?</p>

Resource 3b An algebraic magic square

Copy, cut out and shuffle up the nine cards below.

Arrange the cards to form a magic square.

$a + 4b$	$2a + b$	$8a + 3b$
$5a + 2b$	$7a + 6b$	$7b$
$6a + 9b$	$4a + 5b$	$3a + 8b$

Resource 3c Addition squares

1 Complete these two addition squares.

+	8	14
3
17

+
97	126	...
...	178	537

What number skills are being practised?

What other mathematical skills are involved?

How would you explain and so justify your solution to someone else?

2 Complete these two addition squares.

+	$2c + 3d$	$8c + 2d$	+
$4c + 5d$	$3f + 4g$	$4f - 3g$...
$3c + d$	$3f - 5g$	$2f - g$

What algebraic skills are being practised?

What other mathematical skills are involved?

How would you explain and so justify your solution to someone else?

Resource 3d Reflections on addition squares

What do you notice about the diagonals in the addition squares on **Resource 3c**?

Prove that this must always be the case for a 2 by 2 addition square.

What are the advantages of moving explicitly from numbers to algebra in activities like these?

Resource 3e More addition squares

Complete these addition squares.

+
...	$3a + 5b$	$5a + 3b$
...	$2a + 7b$	$4a + 5b$

+
...	$2g + 5h$	$6g - 3h$
...	$4h - g$	$3g - 4h$

+
...	$8t - 3u$	$5t - 4u$
...	$15t - u$	$2t + 6u$

+
...	$5a^2 + 8ab$	$6a^2 - 3ab$
...	$2a^2 + 10ab$	$3a^2 - ab$

When you have worked on each of the puzzles for a few minutes, make some notes on your answers to the questions on the next page.

How many solutions are there to each of the addition square puzzles?

Prove your statements about the number of solutions to each puzzle.

Resource 3f Approaches to algebra in the classroom

Consider the examples of activities that you have tried out during your study of this module, and the examples that you have looked at in the *Framework for teaching mathematics: Years 7, 8 and 9*.

What objectives from the yearly teaching programmes, and for which year groups, do the activities in this module address?

What other activities from the supplement of examples could you incorporate in lessons to teach these objectives?

How could you adapt or extend the activities for other Key Stage 3 classes?

Consider the questions in this module that guided you through the activities and helped you to reflect on them. Look back through the module and identify the questions that you could incorporate into your questioning of pupils.

How would you introduce activities like these into your classroom? What modifications, if any, would you need to make to your planning, questioning styles or classroom organisation?

Resource 3g Summary and further action on Module 3

Look back over the notes you have made during this module. Identify the most important things to consider and modify in your planning and teaching of algebra.

List two or three key points that you have learned.

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List two or three points to follow up in further study.

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List two or three modifications that you will make to your planning or teaching of algebra.

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List the most important points that you want to discuss with your head of department, or any further actions you will take as a result of completing this module.

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