MODULE

5

Geometrical reasoning 1

OBJECTIVES

This module is for study by an individual teacher or group of teachers. It:

- looks at approaches to developing pupils' visualisation and geometrical reasoning skills;
- considers progression towards geometric proof.

CONTENT

The module is in five parts.

1 Introduction

2 Conventions, definitions and derived properties

3 Deriving properties

4 Looking at a lesson on geometrical reasoning

5 Summary

RESOURCES

Essential

- Your personal file for inserting resource sheets and making notes as you work through the activities in this module
- The *Framework for teaching mathematics: Years 7, 8 and 9*
- Video sequence 3, a Year 8 geometry lesson, from the CD-ROM accompanying this module
- The resource sheets at the end of this module:

  5a Visualisation activities

  5b Conventions and definitions

  5c Deriving properties

  5d More derivations

  5e Bola's lesson

  5f Examples from National Curriculum tests for Key Stages 2 and 3

  5g Summary and further action on Module 5

Desirable

- *Year 9 geometrical reasoning: mini-pack*
  http://www.standards.dfes.gov.uk/keystage3/respub/ma_intery9geom
- *Teaching and learning geometry 11–19*, a joint report from the Royal Society and the Joint Mathematical Council, available at:
  http://www.royalsoc.ac.uk/downloaddoc.asp?id=1191
- The QCA Mathematics glossary for teachers in Key Stages 1 to 4, available at
  http://www.qca.org.uk/279_2104.html

STUDY TIME

Allow approximately 90 minutes.

Part 1 Introduction

1 As pupils move through Key Stages 1 and 2, they progress from a mainly practical and experiential approach to shape and space to a more structured, formalised approach.

The main aim in Key Stage 3 is for pupils to develop their knowledge of geometrical ideas and use it to support geometrical reasoning. This approach lays the foundations for a more formal approach to geometrical proof in Key Stage 4. (A mathematical proof involves establishing the truth of a statement by rigorous logical argument.)

Geometrical reasoning and proof have had greater emphasis in the National Curriculum since 2000 than previously. This emphasis is reflected in the *Framework for teaching mathematics: Years 7, 8 and 9*.

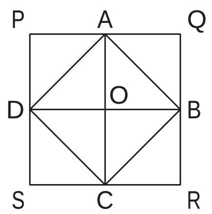
Geometrical reasoning is the focus for this and the next module.

2 Try the two activities on **Resource 5a, Visualisation activities**.

3 The two activities on Resource 5a aim to develop visualisation, geometrical reasoning and justification. These and similar activities make useful oral and mental starters for geometry lessons. Since there is often more than one way of justifying the result, you can ask pupils to compare their justifications by describing them to a partner or to the whole class.

Compare your arguments with those below.

Midpoints



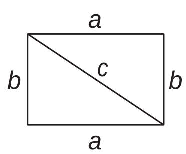
ABCD is a square with an area that is half the area of square PQRS.

A justification:

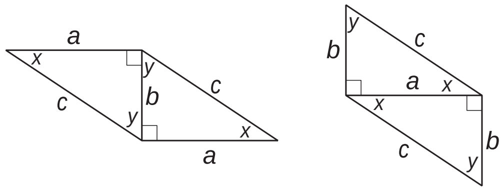
AC and BD are the perpendicular bisectors of the sides of the square PQRS. They are equal in length and bisect each other at right angles. Since AC and BD are the diagonals of ABCD, it follows that ABCD is a square.

The eight triangles formed by the construction lines are congruent and so equal in area (two sides and the included right angle are equal). It follows that the area of square ABCD, which is formed from four of the triangles, is half the area of square PQRS.

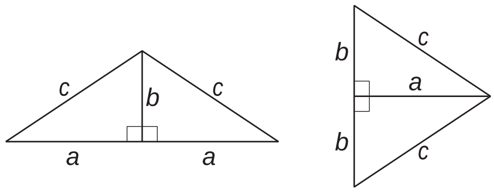
Changing shapes



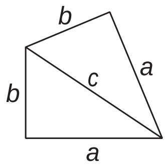
The diagonal of the rectangle creates two identical right-angled triangles. These can be used to make five possible shapes in addition to the original rectangle: two different parallelograms, two different isosceles triangles and a kite.



In these two shapes, it can be shown that opposite sides are equal and opposite angles are equal, making them parallelograms.



In these two shapes, it can be shown that two sides are equal, and that the third side is a straight line, making them isosceles triangles. (It is necessary to prove that the third side is a straight line, otherwise the shape would be a quadrilateral with two adjacent sides equal.)



In this shape, it can be shown that two pairs of adjacent sides are equal, making it a kite.

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4 Now look in more detail at aspects of geometrical reasoning in the *Framework for teaching mathematics: Years 7, 8 and 9*. Study the teaching programmes for Years 7, 8 and 9, Framework section 3, pages 7, 9 and 11.

As you study the teaching programmes, jot down in your personal file some of the words that illustrate the emphasis on geometrical reasoning.

As well as several references to 'explaining reasoning', there are objectives in the Year 9 teaching programme, and in the Year 9 extension programme, which rely on fairly sophisticated thought, such as:

- distinguish between conventions, definitions and derived properties;
- distinguish between practical demonstration and proof (Year 9 extension).

This module focuses in particular on the progression through Key Stage 3 that builds up to these objectives in Year 9.

Part 2 Conventions, definitions and derived properties

1 Study the examples of conventions and definitions on **Resource 5b, Conventions and definitions**. Are you familiar with these?

Other properties of angles and shapes can be derived from these definitions. For example, it is possible to prove, rather than define, that 'vertically opposite angles are equal' or that 'alternate angles are equal'.

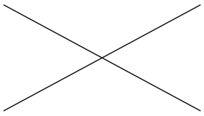
2 Sue Waring, in her book *Can you prove it?*, published by The Mathematical Association, identifies four possible stages for pupils as they work towards a formal proof:

Stage 1 Convince yourself (mental justification)
Stage 2 Convince a friend (oral justification)
Stage 3 Convince a pen-friend (informal written justification)
Stage 4 Convince your mathematics teacher (more formal written justification)

These four stages are illustrated below in relation to a proof that 'vertically opposite angles are equal'.

Stage 1

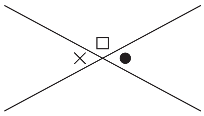
Stage 1 involves convincing yourself.



For example, you might think: 'Those two angles are on a straight line and so are those two angles. So I can take that big angle away from both those straight angles and the two remaining little angles must be equal.'

Stage 2

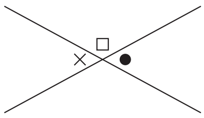
Stage 2 involves convincing a friend.



For example, you might say: 'The angle marked with a circle and the angle marked with a square add up to 180°. The same is true for the angle marked with a cross and the one marked with the square. So this angle (point to the circle) must equal this one (point to the cross).'

Stage 3

Stage 3 involves an informal written justification, which might go like this.



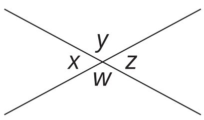
● + □ = 180

× + □ = 180

So × = ●

Stage 4

Stage 4 involves a formal written justification, which might go like this.



$x + y = 180$ (angles on a straight line)

$y + z = 180$ (angles on a straight line)

$\therefore\ x + y = y + z$, giving $x = z$.

3 Now study the examples in the supplement of examples, Framework section 4, pages 178–179.

As you look at the examples, think about the differences between conventions, definitions and derived properties. Distinguishing between a demonstration and a proof is in the Key Stage 4 programme of study and is in the Year 9 extension teaching programme exemplified, in italics, in the last paragraph of page 179.

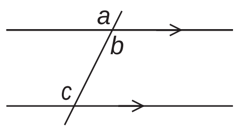
Now study the examples on pages 180–201. These illustrate the progression from informal explanation and justification to formal proof. As you work through the examples, make a note in your personal file of any that would be useful to explore with the classes that you teach.

Part 3 Deriving properties

1 Do the problems on **Resource 5c, Deriving properties**. You are given some definitions of geometric properties and, from these, must deduce some further geometric properties. Aim to produce a stage 4 formal proof wherever possible.

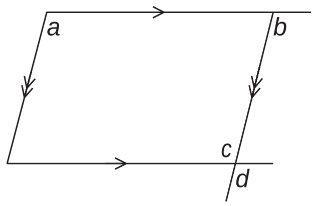
2 Here are some possible arguments that can be used for the problems on Resource 5c.

1



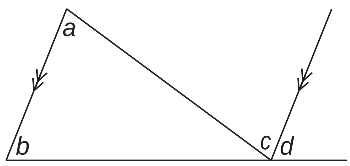
Take any pair of alternate angles, for example a and b.
$a = b$ (vertically opposite angles)
$a = c$ (corresponding angles)
$\therefore\ b = c$
So alternate angles between parallel lines are equal.

2



Take any pair of opposite angles, for example a and c.
$a = b$ (corresponding angles)
$b = c$ (alternate angles)
$\therefore\ a = c$
So the opposite angles of a parallelogram are equal.

3



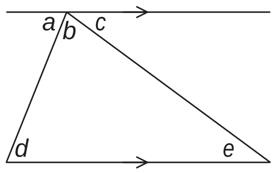
Extend one side of the triangle and construct a line through one vertex parallel to the opposite side, as shown.
$a = c$ (alternate angles)

$b = d$ (corresponding angles)

$\therefore\ a + b = c + d$

So the exterior angle of a triangle equals the sum of the interior opposite angles.

4



$d = a$ (alternate angles)

$e = c$ (alternate angles)

$\therefore\ d + b + e = a + b + c = 180$ (angles on a straight line)

So the sum of the interior angles of a triangle is 180°.

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3 Now try the problems on **Resource 5d, More derivations**.

To find the sum S of the interior angles of an n-sided polygon, identify a point O inside the polygon. Join O to each of the vertices of the polygon, forming n triangles. S can be regarded as the sum T of the angles of all n triangles, less the sum A of the angles around point O. Since A is 360° or 4 right angles, S can be calculated in two ways:

- since each of the n triangles has an angle sum of 180°, T is $180n°$, and $S = 180n - 360$ degrees, which is $(n - 2) \times 180°$;
- since each of the n triangles has an angle sum of 2 right angles, T is $2n$ right angles, and $S = 2n - 4$ right angles.

Part 4 Looking at a lesson on geometrical reasoning

1 In this part of the module, you will have a chance to consider the use of ICT in developing geometrical reasoning.

First study the examples on pages 180–181 of the supplement of examples, Framework section 4. These examples refer to the use of acetate sheets for an overhead projector or use of computer software. As you study the examples, think about the relative advantages of one medium over the other. Note in your personal file any examples that would be useful to explore with the classes that you teach.

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2 Get ready to watch **Video sequence 3, a Year 8 geometry lesson**. The teacher is Bola.

Bola is using dynamic geometry software to begin to develop her pupils' ideas of proof. In the first part of the video sequence, Bola discusses with the class how to name angles and demonstrates the equality of vertically opposite angles. In the second part, which is much later in the lesson, she questions pupils about their proof that the sum of the angles of a triangle is 180°.

As you watch the video, consider the usefulness of dynamic geometry software, focusing on the questions on **Resource 5e, Bola's lesson**.

The video sequence lasts about 5 minutes.

When you have finished watching, spend a few minutes completing the notes you have made on Resource 5e. Then think about how Bola's approach compares with what you would have done.

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3 Study the geometric problems in the examples on using and applying mathematics in the supplement of examples, Framework section 4, pages 14–17. These include examples that make use of computer software or acetate sheets. They also include more examples in which properties of shapes have to be derived by geometrical reasoning.

Add to the notes in your personal file one or more problems that would be useful to offer to the classes that you teach.

Part 5 Summary

1 It is important that Key Stage 3 pupils appreciate the differences between geometric conventions, definitions and derived properties.

As they use and apply their developing knowledge of geometrical properties, pupils in Key Stage 3 should move from informal justifications of their arguments to more formal written proofs.

2 Look at **Resource 5f, Examples from National Curriculum tests**. What definitions would pupils need to know in order to answer the questions?

For each question, think about the kinds of informal or formal arguments that you would expect pupils to give to justify their reasoning.

3 Look back over the notes you have made during this module. Have you identified what you may need to consider and adopt in your planning and teaching of geometry?

Use **Resource 5g, Summary and further action on Module 5**, to list key points you have learned, points to follow up in further study, modifications you will make to your planning or teaching, and points to discuss with your head of department.

4 You may find it interesting to read the article by Paul Chambers on 'Teaching Pythagoras' theorem' from the September 1999 issue of *Mathematics in School*, published by The Mathematical Association, 259 London Road, Leicester LE2 3BE.

If you are interested in reading more about the teaching of geometry in secondary schools, read *Teaching and learning geometry 11–19*, a joint report from the Royal Society and the Joint Mathematical Council. This report reiterates the centrality of geometry to the mathematics curriculum and how important it is that this branch of the subject should not be neglected. Appendix 9: Proof – 'why and what?' is of particular interest. You can download the report from http://www.royalsoc.ac.uk/downloaddoc.asp?id=1191.

You could also download and look at the *Year 9 geometrical reasoning: mini-pack* from http://www.standards.dfes.gov.uk/keystage3/respub/ma_intery9geom.

Resource 5a Visualisation activities

The first part of each of these activities should be carried out without any drawing.

1 MIDPOINTS

Imagine a square.

Join the midpoints of each pair of adjacent sides.

What is the inscribed shape?

How does the area of the inscribed shape relate to the area of the original square?

Now justify your reasoning. Draw a sketch and use informal language if you wish.

[continued on the next page]

2 CHANGING SHAPES

Imagine a (non-square) rectangle.

Cut it along one of its diagonals so that you have two shapes. Call these shapes A and B.

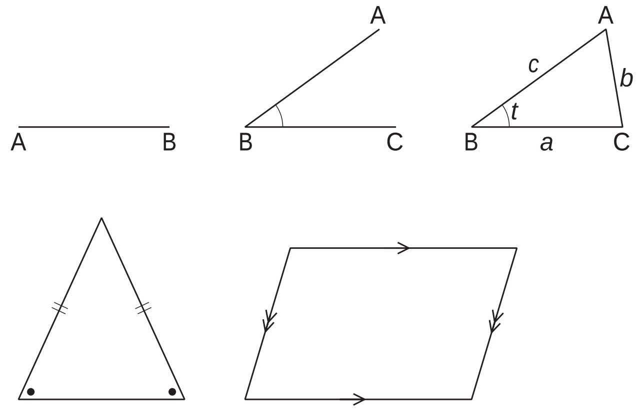
Visualise the different shapes you can make from A and B by matching sides of the same length.

Now sketch each of your new shapes and write its name. For each shape, state the geometrical facts you are using to justify your answer.

Resource 5b Conventions and definitions

CONVENTIONS

Labelling

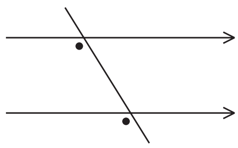


Notation

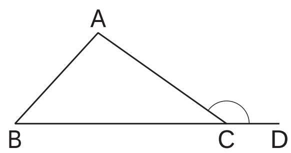
△ triangle ∠ angle ∴ therefore // is parallel to

DEFINITIONS

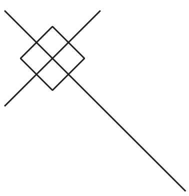
Corresponding angles lie on the same side of a transversal and on corresponding sides of a pair of lines. If the two lines are parallel, the corresponding angles are equal.



An **exterior angle** of a polygon is the angle between a side and an extension of an adjacent side. In this example, ∠ACD is an exterior angle of △ABC.



Perpendicular lines intersect at right angles.



Resource 5c Deriving properties

You are given the following facts:

- the angles on a straight line are supplementary, i.e. they add up to 180°;
- corresponding angles are equal;
- vertically opposite angles are equal;
- opposite sides of a parallelogram are parallel.

Use some or all of these facts, and constructions where necessary, to prove the following in the order 1, 2, 3, 4.

1 Alternate angles are equal.

2 Opposite angles of a parallelogram are equal.

3 The exterior angle of a triangle is equal to the sum of the interior opposites.

4 The angles of a triangle add up to 180°.

Resource 5d More derivations

Look at the Year 9 examples on page 183 of the supplement of examples, Framework section 4.

Derive the formula for the sum of the internal angles of an n-sided polygon as $(n - 2) \times 180°$.

Think of an alternative argument that would lead to '$(2n - 4)$ right angles'.

Resource 5e Bola's lesson

While watching the short video extracts of Bola working with her Year 8 class, consider and make notes on the questions below.

How does the dynamic geometry software facilitate demonstration of a given fact?

How does the dynamic geometry software facilitate proof of a geometrical property?

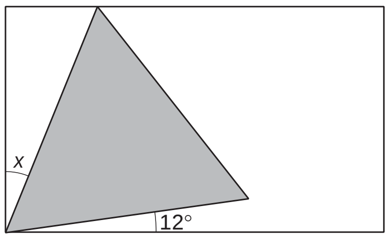
What other aspects of geometrical reasoning could be enhanced by giving pupils the opportunity to move lines and shapes in this way?

Resource 5f Examples from National Curriculum tests

These examples are taken from the National Curriculum tests for Key Stages 2 and 3.

LEVEL 5

1 Here is an equilateral triangle inside a rectangle.

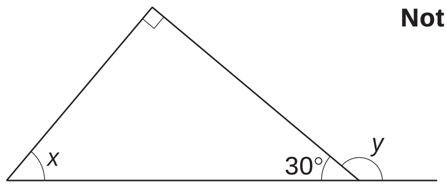


Not to scale

Calculate the value of angle *x*.
Show your working.

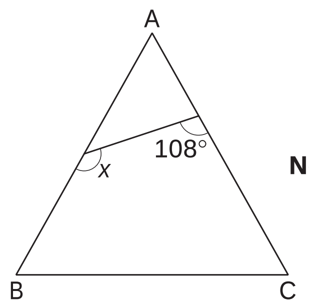
2 Look at this diagram.

Not to scale



Calculate the size of angle *x* and angle *y*.
Show your working.

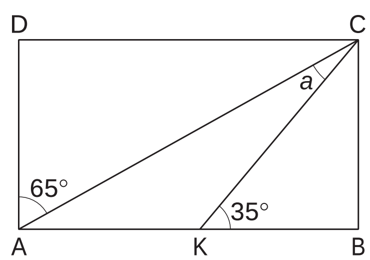
3 Triangle ABC is equilateral.



Not to scale

Calculate the size of angle *x*.
Show your working.

4 The diagram shows a rectangle.

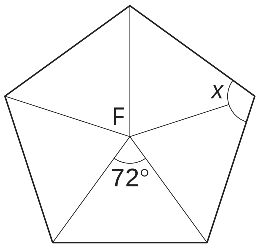


Not drawn accurately

Work out the size of angle *a*.
Show your working.

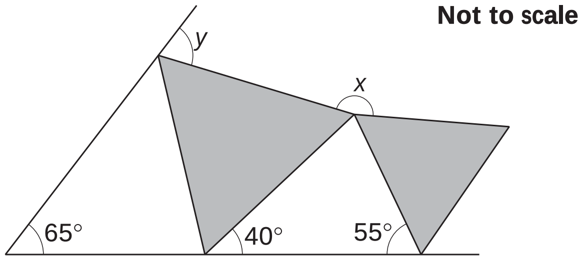
LEVEL 6

1 F is the centre of a regular pentagon.



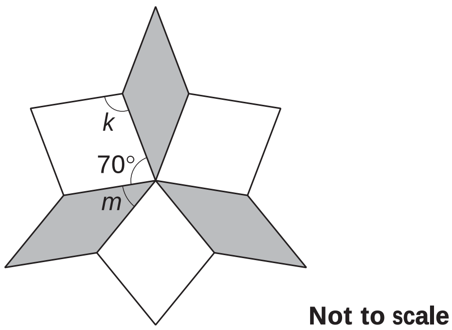
Work out the value of angle *x*.
Explain how you worked out your answer.

2 The diagram shows two shaded equilateral triangles.



Calculate the size of angle *x* and the size of angle *y*.

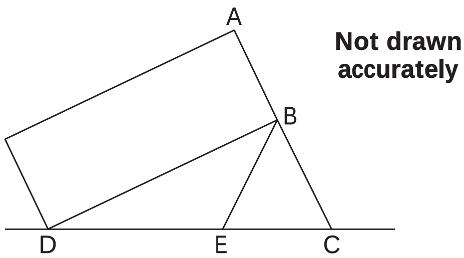
3 The shape below has three identical white tiles and three identical grey tiles.
The sides of each tile are all the same length.
Opposite sides of each tile are parallel.
One of the angles is 70°.



Calculate the size of angle *k* and angle *m*.
Show your working.

LEVEL 7

1 A rectangle just touches an equilateral triangle so that ABC is a straight line.



Show that triangle BDE is isosceles.

Resource 5g Summary and further action on Module 5

Look back over the notes you have made during this module. Identify the most important things to consider and modify in your planning and teaching of geometry.

List two or three key points that you have learned.

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List two or three points to follow up in further study.

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List two or three modifications that you will make to your planning or teaching of geometry.

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List the most important points that you want to discuss with your head of department, or any further actions you will take as a result of completing this module.

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