MODULE

1

Approaches to calculation

OBJECTIVES

This module is for study by an individual teacher or group of teachers. It:

- discusses the development of calculation strategies from Key Stage 2 to Key Stage 3 to build on pupils' knowledge, skills and understanding;
- considers teaching methods to support the development of calculation strategies, using subtraction as a focus.

CONTENT

The module is in five parts.

1 Mental calculation

2 The use of jottings in mental calculation, with special reference to subtraction

3 Written calculation

4 Teaching subtraction in Year 7

5 Summary

RESOURCES

Essential

- Your personal file for inserting resource sheets and making notes as you work through the activities in this module
- The *Framework for teaching mathematics: Years 7, 8 and 9* (DfES 0020/2001)
- The resource sheets at the end of this module:
  - 1a Teaching mental calculation
  - 1b Approaches to mental calculation
  - 1c Extending pupils' mental skills
  - 1d Using a blank number line to support subtraction
  - 1e Teaching written calculation
  - 1f Summary and further action on Module 1
- Video sequence 1, a Year 7 subtraction lesson, from the CD-ROM accompanying this module
- Section 6, the supplement of examples for Years 4, 5 and 6, from the *Framework for teaching mathematics from Reception to Year 6* (DfES ref: NNFT)
  If you don't have a copy this document, you can download section 6 from http://www.standards.dfes.gov.uk/primary/publications/mathematics/math_framework/teachingprogrammes/year6/55905

Desirable

- *Teaching written calculations: guidance for teachers at Key Stages 1 and 2*
  This publication (reference QCA/99/486) can be ordered at £3 per copy from QCA Publications, PO Box 99, Sudbury, Suffolk CO10 2SN (tel: 01787 884444).

STUDY TIME

Allow approximately 90 minutes.

Part 1 Mental calculation

1 This module discusses calculation methods. During your study, you will consider mental and written methods of calculation, and will watch a lesson that focuses on the teaching of subtraction. The use of the calculator is discussed in Module 2.

2 Mathematics teachers in Key Stage 3 have noticed a difference in the calculation skills of pupils transferring from Key Stage 2 in the last few years. Take a moment to consider what differences you and your colleagues have noticed in the standards of pupils' calculating skills.

Would you agree that Key Stage 3 pupils in your school:

- are more confident in calculating mentally;
- have a wider range of mental calculation strategies;
- are more willing and able to describe their calculation methods?

3 Read **Resource 1a, Teaching mental calculation**. This short article describes the mental calculation skills and opportunities to learn that pupils need to develop and experience throughout Key Stage 3.

Allocate a page of your personal file on which you can note points to discuss later with your head of department. Now think about the article you have just read. To what extent do the skills and opportunities described correspond with your experiences as a teacher of mathematics? Jot down up to three main differences for discussion with your head of department.

4 Try the calculations on **Resource 1b, Approaches to mental calculation**.

Think about the knowledge and understanding that underpins each calculation. For nearly all of the calculations, shown again below, you should expect the majority of Year 7 pupils to calculate the answers in their heads.

How do these methods differ from your methods?

$19 \times 6 = 114$	Find 20×6 then subtract 6 (mental method)
$67 \times 7 = 469$	Find 60×7 and 7×7 and add the results (may need informal jotting)
$85.5 \div 10 = 8.55$	Shift digits one place to the right (mental method which depends on multiplying and dividing by powers of 10, an important skill)
5% of 84 = 4.2	Find 10% then halve the answer (mental method)
$5.2 \times 20 = 104$	Multiply 5.2 by 10 then double (mental method)
$14.2 \div 4 = 3.55$	Halve then halve again (mental method, possibly with an intermediate note of 7.1 which is harder to halve)
$0.12 \times 0.6 = 0.072$	Multiply 12 by 6 then adjust (needs secure knowledge of relative size of numbers)
¾ of 56 = 42	Find ¼ of 56 then multiply by 3 (mental method)

$\sqrt[3]{64} = 4$	Use knowledge of powers (i.e. know that $4^3 = 64$) or use trial and improvement

5 The *Framework for teaching mathematics: Years 7, 8 and 9* provides an overview of the progression in mental calculation skills in Key Stage 3. Study the objectives for mental calculation strategies, mental methods and rapid recall of number facts in the yearly teaching programmes for Years 5 to 9 (see the Key Stage 3 Framework section 3) and the related examples (see section 4, pages 88–103).

To what extent do these objectives correspond with your experiences as a teacher of mathematics? If there are differences, add up to three points to your notes to discuss later with your head of department.

6 Schools with Key Stage 3 pupils have already begun to ensure that pupils' mental skills are kept sharp and are developed in Key Stage 3.

Consider the three questions on **Resource 1c, Extending pupils' mental skills**. Make some notes in your personal file on what you might need to do to build on what is already in place and working well.

7 Now compare the notes you have made on Resource 1c with the suggestions below. Refine or add to your notes if you see relevant ideas.

Planning and teaching

- Include regular practice of mental skills in your teaching plans.
- Build the teaching of mental calculation strategies into your teaching plans.
- Plan for progression throughout the key stage – not just in units involving number and calculation.
- Expect pupils to use mental methods as a *first* resort – always asking themselves: 'Can I do this in my head?'
- Provide plenty of opportunities for pupils to explain, discuss and evaluate their mental calculation strategies and those of others.

Integrating mental calculation into starters and main teaching activities

- Use oral and mental starters for practising skills already taught.
- Use the main part of lesson to teach skills, explaining and discussing different methods for more complex calculations.
- Build in questions that can be answered mentally when pupils are working on different topics – for example, on averages or volume or algebraic work.
- Oral and mental starters can be used to rehearse calculation skills needed in the main part of lesson.

Ensuring suitable differentiation

- Target questions carefully to ensure the involvement of all pupils, whether in mixed-ability or setted groups.
- Use practical resources to ensure maximum participation – for example, small whiteboards, number cards or fans.
- Use resources such as number lines, number grids and place value charts to support pupils where appropriate.
- During the main part of the lesson:

- group pupils so that you can directly teach those with a particular need;
- have ready some non-routine, challenging problems to extend the most able.

- If possible, use a teaching assistant to offer targeted support to an identified group.

Part 2 The use of jottings in mental calculation, with special reference to subtraction

1 As calculations become more complex, it can be hard for pupils to hold all the intermediate steps of a calculation in their heads. At this stage, recording some or all of the steps as informal jottings becomes part of the mental strategy.

To illustrate the ideas involved, we will focus on subtraction, and in particular on the use of a blank (or empty) number line.

Read **Resource 1d, Using a blank number line to support subtraction**.

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2 In your personal file, try some examples of subtraction calculations using a blank number line approach. For example:

- use 'counting back' to work out 435 – 178;
- use 'counting down' from the larger to the smaller number to find the difference between 435 and 178;
- use 'counting up' from the smaller number to the larger number to find how many more than 178 is 435.

Which of these three methods did you prefer?

Now try working out 435 – 178 by translating it to an easier calculation.

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3 To what extent do you agree with these statements?

The blank number line (BNL):

- deepens pupils' understanding of numbers and operations;
  The BNL is a powerful way of modelling operations on numbers. It deepens pupils' understanding of the number system by helping them to focus on the characteristics of numbers (their absolute size, their relative size, their proximity to 'friendly' numbers, and so on), and not just the operation.
- helps pupils to develop and refine their mental methods;
  The BNL helps pupils to construct mental approaches based securely on their level of understanding. It can also help them to refine their existing mental methods and extend their methods to numbers that are more difficult.
- makes a useful diagrammatic record of calculation strategies;
  Pupils can use the BNL to help explain to themselves and others how they have tackled a calculation. It stimulates explanation and discussion of calculation strategies.
- can be extended to calculations involving negative numbers and decimals.
  As such, the BNL is a tool that should remain familiar to all pupils and not be seen merely as a support tool for those with poor calculation skills.

Part 3 Written calculation

1 Read **Resource 1e, Teaching written calculations.**

2 The *Framework for teaching mathematics from Reception to Year 6*, which should be readily available in all secondary schools, provides valuable guidance on the progression to written calculations. Look at pages 48–51 of section 6, the supplement of examples for Years 4, 5 and 6.

Find the objective: 'Develop and refine written methods for subtraction, building on mental methods'. Consider progression across Years 4, 5 and 6 by looking at the headings in each column: informal written methods; standard written methods; extend to decimals. Track the progression through the years by scanning across the columns.

Now look at pages 66–69 of section 6, which illustrate progression in multiplication and division.

Part 4 Teaching subtraction in Year 7

1 Consider this subtraction.

$$\begin{array}{r} 1000 \\ -\ 99 \\ \hline \end{array}$$

- How would you expect Year 7 pupils in you school to tackle this question?
- What would you do next with pupils who attempted the standard method of decomposition but got it wrong?

In your personal file, jot down your thoughts on these two issues before continuing.

2 Watch **Video sequence 1, a Year 7 subtraction lesson**. The lesson is taught by Catherine. All pupils in the class achieved either level 3 or level 4 in the Key Stage 2 National Curriculum tests. In a previous lesson, Catherine gave pupils a short test to assess their competence in using written methods of subtraction for whole numbers.

The video sequence lasts about 13 minutes.

When you have finished watching, spend a few minutes considering how Catherine's approach to tackling her pupils' difficulties with subtraction compares with what you would have done.

3 Read the notes below. These give some background information on the teaching and learning activities that you have seen in the video.

- Many pupils believe that because a subtraction is presented vertically it must be tackled using decomposition. For this reason, questions such as 1000 – 99 tend to be presented horizontally in the Key Stage 2 and Key Stage 3 National Curriculum tests.
- Pupils who struggle with decomposition might be able to tackle 1000 – 99 in their heads without any writing.
- A blank number line provides useful support for pupils' informal methods.
- The blank number line is only one of several possible models that can help to reinforce pupils' understanding of subtraction.

- Giving pupils opportunities to work in pairs to discuss calculation strategies can deepen their understanding. Observing this kind of discussion gives teachers valuable evidence of pupils' thinking.
- The teaching of the decomposition method of subtraction needs to stress the place value of digits. For example, in changing:

  85 to $^{7}\not{8}^{1}5$,

  say 'eighty and five becomes seventy and fifteen' rather than 'eight becomes seven with a little one beside the five'.

Part 5 Summary

1 Some important principles in the teaching of calculation strategies are:

- Allow pupils to show what they know and can do.
  Establish pupils' competences before trying to move them forward. Effective assessment ensures that pupils are allowed to show what they know and can do. Don't rely solely on narrow tests of competency in particular written methods.
- Plan to maintain and develop mental calculation strategies.
  Maintain and extend pupils' repertoire of mental approaches throughout Key Stage 3 through regular planned practice and focused teaching.
- Value and build on informal jottings.
  Informal jottings (for example, using a blank number line) are legitimate support for harder mental calculations. They should be valued and used as the basis of discussions, not hidden on scraps of paper or in the backs of exercise books!
- If pupils consistently struggle with a calculation method, backtrack to an earlier stage to establish firm understanding.
  For pupils who are not secure with formal written approaches, it is important to go back to building confidence with informal methods that reflect their level of understanding.
- Teach pupils to discuss their own and others' strategies.
  Encourage pupils to talk through their approaches, paying attention to the necessary vocabulary. This allows them to exchange and evaluate different approaches. The discussion also provides teachers with valuable insights into pupils' understanding.
- Emphasise 'doing it in your head if possible' and 'checking your answer'.
  Remind pupils constantly to ask: 'Can I do this in my head?', 'How can I check my answer?' and 'Does my answer make sense?'

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2 Look back over the notes you have made during this module. Have you identified the most important things that you may need to consider and adopt in your planning and teaching of calculations?

Use **Resource 1f, Summary and further action on Module 1**, to list key points you have learned, points to follow up in further study, modifications you will make to your planning or teaching, and the main points to discuss with your head of department.

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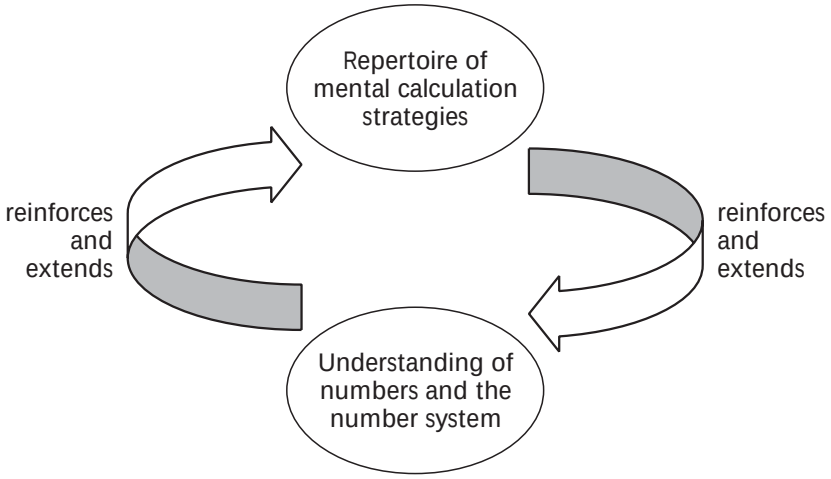
3 You will find useful guidance on teaching written calculations in *Teaching written calculations: guidance for teachers at Key Stages 1 and 2* (see the first page of this module for ordering information).

Resource 1a Teaching mental calculation

The ability to calculate mentally lies at the heart of numeracy. The diagram shows the relationship between two elements of pupils' learning of mathematics: the development of calculation strategies (particularly mental strategies) and their understanding of numbers and the number system. Each element supports and depends on the other, so teachers need to plan carefully to address both elements in tandem in their teaching.



A number of skills are involved that pupils need to practise and extend throughout Key Stage 3. These skills are:

- recognising that mental approaches should be considered as a first resort;
  i.e. remembering to ask: 'Can I do this in my head, without any writing?' and, if not: 'Can I do it in my head with some jottings to help?'
- remembering number facts and recalling them without hesitation;
  e.g. recalling facts such as multiplication and division facts, the percentage and decimal equivalents for common fractions, squares and cubes, …
- using known facts to figure out new facts;
  e.g. knowing that if 3 × 6 = 18 then 3 × 0.6 = 1.8 and 300 × 60 = 18 000.
- drawing on a repertoire of mental strategies to work out calculations, with some thinking time, and being able to explain methods;
  i.e. being able to call on a range of strategies and choose strategies appropriate for the calculation.
- understanding and using relationships between the four operations to find answers and check results;
  e.g. knowing that if 20 × 36 = 720 then 720 ÷ 20 = 36.
- approximating calculations to judge whether an answer is reasonable;
  e.g. knowing that 490 ÷ 24 will be roughly the same as 500 ÷ 25, i.e. approximately 20, and use this to check that the answer is sensible.
- solving problems, including identifying what operations to use and the steps to take.

Teachers need to make sure that pupils have plenty of opportunities to:

- strengthen their understanding of the number system, including place value, decimal partitioning, and so on;
- describe and explain their mental calculation methods;

Decimal partitioning

3.478
= 3 + 0.4 + 0.07 + 0.008
= 3 + $^{4}/_{10}$ + $^{7}/_{100}$ + $^{8}/_{1000}$

- appreciate that different methods will work for a particular calculation, and to discuss which are the most efficient;
- deepen their awareness of how the laws of arithmetic apply to mental calculations, and apply these laws in the context of algebra;
- continue to practise the mental calculation skills they have already developed in Key Stage 2;
- extend their mental skills to cope with more complex calculations, including those involving decimals, fractions, percentages, powers, conversions of measurements, and so on.

Laws of arithmetic

commutative law

$8 + 47 = 47 + 8$
$15 \times 36 = 36 \times 15$

associative law

$13 + (7 + 8) = (13 + 7) + 8$
$8 \times (5 \times 9) = (8 \times 5) \times 9$

distributive law

$(20 \pm 3) \times 7$
$= (20 \times 7) \pm (3 \times 7)$

$(60 \pm 9) \div 3$
$= (60 \div 3) \pm (9 \div 3)$

Resource 1b Approaches to mental calculation

Do the following calculations mentally, then make a note of your method.

19×6

67×7

$85.5 \div 10$

5% of 84

5.2×20

$14.2 \div 4$

0.12×0.6

$\frac{3}{4}$ of 56

$\sqrt[3]{64}$

Which calculations would you *expect* pupils to attempt mentally?

What kinds of jotting help to support pupils' mental methods?

Resource 1c Extending pupils' mental skills

Focus on each of these three questions in turn. For each question, jot down three things that you could do **to build on and extend pupils' mental skills**.

How could you strengthen your planning or teaching of mental skills in mathematics lessons?

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How could you further the ways in which you integrate mental calculation into both the oral and mental starter and the main teaching activities?

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How could you strengthen the ways in which you provide for differentiation?

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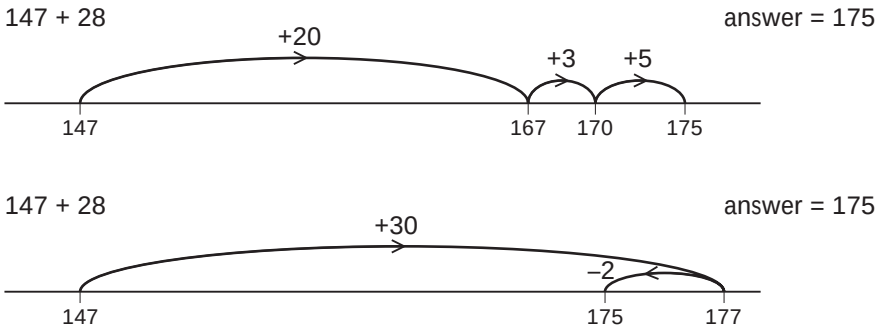
Resource 1d Using a blank number line to support subtraction

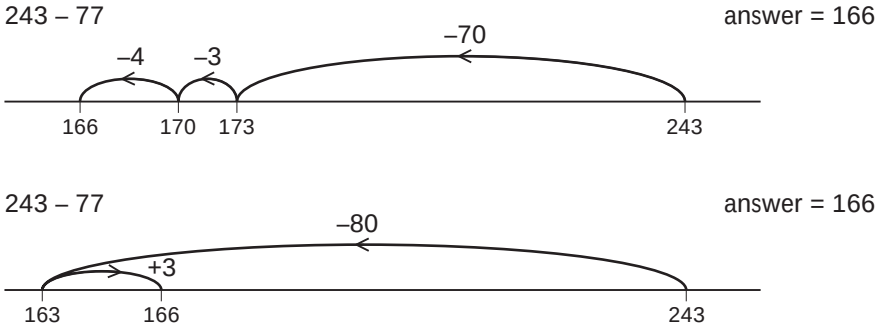
Recent research has shown that the blank (or empty) number line can be a powerful model for building understanding of and developing calculation strategies.

First, here are two examples of the blank number line representing addition.



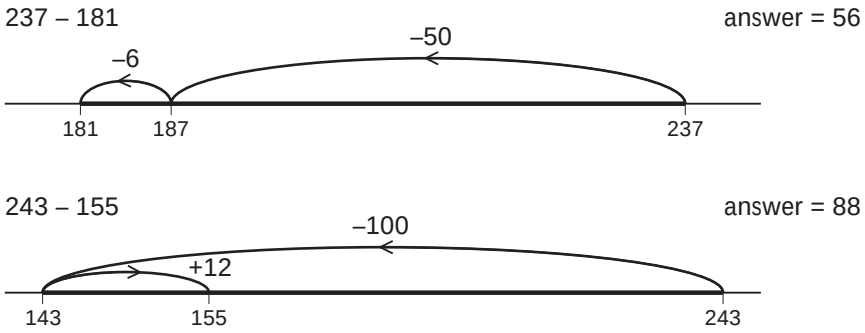
Both of these model common mental approaches. In each case, we start by marking the first number (or, if using a more sophisticated approach, the smaller number). We then count *on from* that number *by* the amount we are adding on. In the second example, the fact that 28 is close to a decade number (30) prompts the compensation approach illustrated. Note that the answer appears as a position on the number line.

One model of subtraction is the opposite of 'adding on' (i.e. subtraction is the inverse of addition). To illustrate this on the number line we count *back from* the first number *by* the amount we are subtracting. Here are two possible representations. Note again that the answer appears as a position on the number line.



An alternative and powerful model for representing subtraction is to see it as the difference between two numbers. Pupils need to be familiar with this view. On the number line, the idea is represented as the length of the line segment between the two numbers (the thicker lines in the examples below). This length can be calculated in different ways. To start, the two numbers whose difference is sought are marked on the line. This action helps focus on the characteristics of the two numbers which, in turn, influences the approach taken.

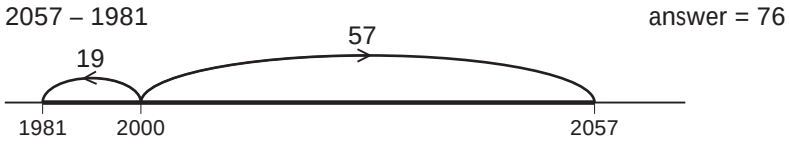
In this first example, the difference is calculated by counting *down* from the larger number *to* the smaller.



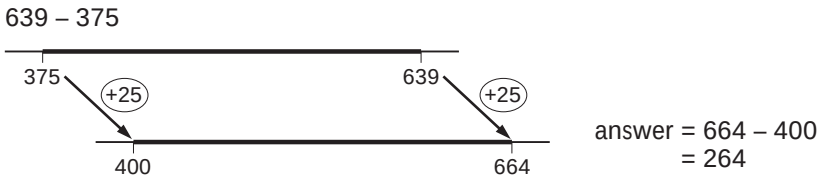
The choice of steps will depend upon how the pupil sees the numbers. Note that the answer is now the gap, not a position on the line.

An alternative approach, clearly equivalent, is to count *up* from the smaller *to* the larger number. This is analogous to a shop assistant (with no electronic till) counting out change. Again, the actual steps will depend upon the numbers involved and pupil preferences.

For some pairs of numbers a useful model is 'jumping out' from an obvious 'bridge point'.



Finally, this picture of difference leads to the idea of translating the pair of numbers up or down the number line without changing the gap between them. In this way, it is possible to translate one subtraction to another with the same answer but involving friendlier numbers. For example, 639 – 375 is equivalent to 664 – 400, which is an easy mental calculation.



Familiarity with this dynamic image can help pupils overcome difficulties they encounter when using compensation methods mentally. For example, using the calculation above, a pupil spotting that 375 is near 400 might begin by subtracting the 400 from 639 to get 239. She now has to compensate for the additional 25 subtracted but may be unsure whether to add it to or subtract it from 239. In order to 'keep the gap the same' she clearly needs to add it.

Resource 1e Teaching written calculation

Nearly all pupils will have been introduced to formal written calculation methods by the end of Key Stage 2. A significant minority of pupils may not have a secure enough understanding of number to cope with some standard methods. The compactness of traditional standard methods, particularly working with digits without referring to their values, prevents some pupils from understanding why a method works. This is especially true of subtraction and long division.

There is evidence from Key Stage 2 tests that pupils can make errors because they try to use a standard method (such as decomposition for subtraction) when they are not secure with it. These pupils would be more successful if they used an informal method which they fully understand and with which they are more confident.

The aim is that pupils choose appropriate calculation strategies that they use accurately, efficiently and with understanding. To achieve this, all teachers of mathematics need to understand the progression from informal mental methods of calculation to efficient, compact written methods for addition, subtraction, multiplication and division. The QCA booklet *Teaching written calculations: guidance for teachers at Key Stages 1 and 2* illustrates this progression. It was written originally for primary teachers but has useful guidance for teaching Key Stage 3 pupils who are not yet consistently competent in using such methods. All secondary schools were given a copy of the booklet in the summer of 2000.

The progression towards standard written methods can be summarised as:

- establishing mental methods, based on good understanding of place value;
- informal recording of mental methods, becoming more structured;
- more formal expanded written methods for addition and subtraction, leading to more compact standard methods;
- informal methods for multiplication and division, built on an understanding of mental strategies;
- standard methods used efficiently and accurately, and with understanding, with increasingly more complex calculations.

Informal recording to support or explain mental calculations

For example:

43

40 + 3

↓ ↓ × 6

240 + 18 = 258

×	40	3
6	240	18

240 + 18 = 258

Some factors that Key Stage 3 teachers need to consider when they are planning and teaching are as follows.

- Pupils should continue to be expected to use mental methods as a first resort, not only in mathematics lessons but also in other subjects.
- Formal written methods for addition and subtraction should be established by Year 6 and used in Key Stage 3 with an increasing range of whole numbers and decimals.
- Multiplication and division methods will need to be developed further. Some pupils may still use informal written methods, such as:
  - a 'grid' method for multiplying two- or three-digit numbers, which can be consolidated and built on in Key Stage 3 and related to work in algebra;
  - a 'chunking' method for division, which will also need to be built on and extended in Key Stage 3.
- With written methods, pupils should be able to use approximate values and mental calculations to estimate the size of the answer for checking purposes.

Resource 1f Summary and further action on Module 1

Look back over the notes you have made during this module. Identify the most important things to consider and modify in your planning and teaching of calculation.

List two or three key points that you have learned.

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List two or three points to follow up in further study.

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List two or three modifications that you will make to your planning or teaching of calculation.

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List the most important points that you want to discuss with your head of department, or any further actions you will take as a result of completing this module.

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