# *Interacting with mathematics in Key Stage 3*

*Year 7 fractions and ratio: mini-pack*

# Contents



# Year 7 fractions and ratio: sample unit

## Introduction

This unit has been prepared on the assumption that it will be the first on the topic in Key Stage 3. As part of the National Numeracy Strategy, there has been an increased emphasis on the teaching of fractions in Key Stages 1 and 2. To assist curriculum continuity, the challenge at the beginning of Key Stage 3 is to draw out pupils' prior knowledge and to build on it – giving pupils a sense that they are reviewing previous work and meeting new challenges.

The unit is a precursor to the Year 8 multiplicative relationships unit and the Year 9 sequel on proportional reasoning. Together they provide a sequence of core units across the key stage, addressing crucial concepts linking multiplication, division, fractions, ratio and proportion. Proportional thinking is required in number, algebra, shape, space and measures, and handling data.

Research shows that making links between interconnected ideas helps pupils to make sense of the subject, avoid misconceptions and retain what they learn. This unit concentrates on making links between:

- fraction operators, multiplication and division;
- equivalent fractions and ratios;
- fractions, decimals and percentages.

As with previous units, this Year 7 unit has been developed through a flexible use of the sample medium-term plans.

- It replaces Number 2 and addresses most of the objectives from that unit.
- It brings forward objectives relating to ratio and proportion, previously started in Number 4.
- Number 4 should be used as an opportunity to consolidate and apply strategies developed in this unit.
- Addition and subtraction of fractions is addressed later in Number 5.

For schools currently using the QCA bridging unit *Linking fractions, decimals and percentages*, this unit provides an alternative sequel to the Year 6 lessons in that pack. Schools that choose to continue with the QCA pack could adapt this unit for later in the year in place of Number 4.

The unit plan is divided into two phases, setting out the stages of development. Teachers should translate these into lessons according to the length of lessons and the response of pupils. For example, it might be appropriate to increase the pace of the main activity or to repeat particular starters.

### Phase 1 (key lesson plus two lessons)

- In the starters, pupils review their knowledge of fractions, decimals, percentages, ratio and proportion from Key Stage 2.
- In the main activities, pupils use images for fractions and ratio to establish connections between operations, symbols and language to:
  - build on their existing knowledge and develop understanding of fractions as operators;
  - link fractions and ratio and consider equivalent expressions.

**Phase 2 (three lessons)**

- In the starters, pupils practise skills of simplifying fractions and converting between fractions, decimals and percentages.
- In the main activities, pupils draw on precursor ideas in phase 1 to:
  - explore multiplicative relationships within a small set of simple fractions/ratios;
  - use knowledge of these particular relationships to solve simple problems by informal methods.

## Differentiation

In planning this unit, consideration was given to the fact that some schools teach pupils in mixed-ability classes rather than in sets. The following features are built into the unit plan to help match the work to different pupil groupings.

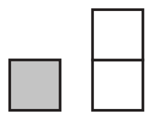
- The first lesson aims to get all pupils to reveal what they know, understand and can do from Key Stage 2 (see pages 10–13 for the key lesson plan).
- Visual images (e.g. fraction diagrams) and practical apparatus (e.g. linking blocks) are used to support the development of mathematical language and notation, in order to engage all pupils in a variety of learning experiences and make meanings clear.
- The emphasis of every lesson is on making links. This should be of particular value to lower attaining pupils, who are less likely to make these links without help. Higher attaining pupils should meet sufficient challenge in clarifying and articulating these connections.
- The numbers involved are easy to handle mentally, keeping the emphasis on understanding and avoiding difficult calculation.
- Some tasks can be presented in an open-ended way (e.g. building up web/spider diagrams, extending or designing similar problems).
- Carefully managed paired and small-group work encourages pupils to discuss and learn from each other and enables tasks to be shared according to pupils' capabilities.

# Unit plan

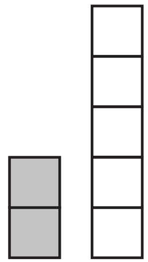
## Phase 1 (about three lessons)

| Oral and mental starter | Main teaching | Key vocabulary |
|---|---|---|
| **Objectives**<br>• Consolidate and **extend mental methods of calculation to include decimals, fractions and percentages**, accompanied where appropriate by suitable jottings.<br>• Understand the relationship between ratio and proportion; use direct proportion in simple contexts. | **Objectives**<br>• Use fraction notation to describe parts of shapes and to express a smaller whole number as a fraction of a larger one; **identify equivalent fractions**.<br>• Calculate simple fractions of quantities (whole-number answers); multiply a fraction by an integer.<br>• Use ratio notation.<br>• Represent problems mathematically, making correct use of symbols, words and diagrams. | operator, multiplier<br>increase, decrease<br>fraction, decimal fraction<br>proper/improper fraction, mixed number<br>numerator, denominator<br>convert, equivalent<br>ratio (*a* : *b*)<br><br>**Resources**<br>• 'Images of fractions':<br>– **either** as an ITP* from the CD-ROM<br>– **or** as OHTs (resource sheets FR1–FR4)<br>• Linking blocks or alternative (e.g. Cuisenaire rods)<br>• Large sheets of plain paper<br>• Mini-whiteboards, calculators (for plenary only)<br><br>**Supplementary notes** (in this mini-pack)<br>• Key lesson (pp. 10–13)<br>• Prompts for phase 1 (pp. 14–19)<br><br>***Interactive teaching programs (ITPs)**<br>ITPs are simple computer programs offering visual images and graphics to support teaching of specific mathematical topics. Instructions are given on the CD-ROM, if required, but most ITPs are easy to use after a few minutes of exploration. |

| Oral and mental starter | Main teaching | Notes | Plenary |
|---|---|---|---|
| First lesson: see plan for extended starter, to review knowledge of fractions, decimals, percentages from KS2.<br><br>Recap on KS2 language of ratio and proportion, including likely images used for solution. Draw on pupils' knowledge choosing pitch and terminology accordingly. For example:<br>There are 5 toffees for every 2 chocolates in a box of sweets.<br>**Q** If there are 15 toffees how many chocolates are there?<br>Model the solution using a number line:<br>5 10 15 20 25<br>2 4 6 8 10<br>**Q** How many sweets altogether?<br>**Q** What proportion of the sweets are toffees?<br>**Q** What proportion of the sweets are chocolates?<br>Further contexts: Framework p. 4. | **Fraction operators**<br>Framework supplement of examples, pp. 60, 64, 66, 68.<br>**Stage 1:** Detailed plan for first lesson (prompts pp. 11–13) on developing images of thirds (use ITP *Images of fractions*, OHTs or sketches):<br>• multiples of one third, e.g. $7 \times \frac{1}{3}$;<br>• thirds of whole numbers, e.g. $\frac{1}{3}$ of 7.<br><br>Establish results such as $\frac{7}{3}$ (seven thirds) = $7 \div 3$ (seven divided by three) and draw web diagram of expressions equivalent to $\frac{7}{3}$ : $7 \times \frac{1}{3}$, $2\frac{1}{3}$, $7 \div 3$, $\frac{1}{3}$ of 7 (see plenary column).<br>Follow a similar sequence to develop images for quarters and fifths.<br><br>**Stage 2:** Develop a sequence of images for $\frac{1}{3}$ of 7, $\frac{2}{3}$ of 7,... , $\frac{6}{3}$ of 7.<br>Establish that, for example:<br>• $\frac{5}{3}$ of 7 = $\frac{5}{3} \times 7$ ('of' means 'multiply');<br>• $\frac{5}{3}$ of 7 = $(7 \div 3) \times 5$<br>• the answer will be an increase in 7 because $\frac{5}{3} > 1$.<br><br>Develop similar sequences for quarters and fifths.<br><br>**Ratios and equivalence**<br>(Prompts p.18) Using linking blocks, pupils set up this arrangement.<br><br>Another way of expressing that the number of black blocks is $\frac{1}{2}$ of the number of white blocks is 'the *ratio* of the number of black blocks to the number of white is *1 to 2*' (no. black : no. white = 1 : 2). Using more blocks in each 'strip', consider other fractions (ratios) equivalent to $\frac{1}{2}$ (1 : 2). Describe this relationship as 'one black for every two white'. Construct a diagram.<br>Repeat for other unit fractions, such as $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$. For each, pupils build up a diagram or table of equivalent ratios, fractions and divisions. | Keep a record of pupils' contributions in the starter to use in final plenary of the unit.<br><br>**Extension/support**<br>This is not delineated here because **all** pupils should benefit from developing a shared image of fractions. Higher attaining pupils will secure their understanding by having to explain connections. For other pupils, some of these connections will be new.<br><br>The unit focuses mainly on halves, thirds, quarters, fifths and associated ratios.<br><br>The blocks illustrate use of fractions to compare any two quantities (not just parts of a whole). Pupils may not have met ratio notation before. | First lesson: Copy a web diagram (see detailed plan), such as:<br>$7 \times \frac{1}{3}$ $2\frac{1}{3}$<br>$\frac{7}{3}$<br>$7 \div 3$ $\frac{1}{3}$ of 7<br><br>Explore changes (e.g. replace $2\frac{1}{3}$ by $2\frac{2}{3}$), asking pupils to say how the other entries will have to change and explain links.<br><br>Remind pupils of the relationship between fractions and division, e.g. $\frac{3}{8} = 3 \div 8$. Use calculators to convert $\frac{3}{8}$ to a decimal. Establish equivalence of $\frac{3}{8} \times 20$ and $0.375 \times 20$ and check with calculator. Try another example, e.g. $\frac{5}{16} \times 20$.<br><br>Start with strips showing a ratio in unsimplified form, e.g. 6 : 18. Ask pupils what fraction this is equivalent to and why. Encourage ways of 'seeing' the result in the strip image. (See prompts p. 19.) |

## Phase 2 (about three lessons)

| Oral and mental starter | Main teaching | Key vocabulary |
|---|---|---|
| **Objectives**<br>• **Simplify fractions and identify equivalent fractions.**<br>• Use ratio notation.<br>• Understand percentage as the 'number of parts per 100'; **recognise the equivalence of percentages, fractions and decimals**. | **Objectives**<br>• Use fraction notation to describe parts of shapes and to express a smaller whole number as a fraction of a larger one; **simplify fractions and identify equivalent fractions**.<br>• Calculate simple fractions of quantities and measurements (whole-number answers); multiply a fraction by an integer.<br>• Understand percentage as the 'number of parts per 100'; **recognise the equivalence of percentages, fractions and decimals;** calculate simple percentages.<br>• Understand the relationship between ratio and proportion; use direct proportion in simple contexts; use ratio notation and divide a quantity into two parts in a given ratio; solve simple problems about ratio and proportion using informal strategies.<br>• Consolidate and **extend mental methods of calculation to include decimals, fractions and percentages**, accompanied where appropriate by suitable jottings; solve simple word problems mentally.<br>• **Check a result by considering whether it is of the right order of magnitude** and by working the problem backwards.<br>• Represent problems mathematically, making correct use of symbols, words and diagrams.<br>• Present and interpret solutions in the context of the original problem; **explain and justify methods and conclusions**, orally and in writing. | operator, multiplier<br>inverse<br>increase, decrease<br>fraction, decimal fraction, percentage (%)<br>proper/improper fraction, mixed number<br>numerator, denominator, reciprocal (optional)<br>convert, equivalent<br>ratio (*a* : *b*)<br>proportion<br><br>**Resources**<br>• Linking blocks or alternative (e.g. Cuisenaire rods); optional extra, ITP *Ratio strips*)<br>• Large sheets of plain paper<br>• Whiteboards<br>• Resource sheets FR5–FR8 (ratio strips)<br>• Sets of problems (resource sheets cut into cards):<br>FR9 problems related to ratios 2 : 5 and 3 : 4<br>FR10 problems related to ratios 4 : 5 and 2 : 3<br>FR11 mixed problems<br>• Prepared posters or OHTs from first lesson (for final plenary)<br><br>**Supplementary notes** (in this mini-pack)<br>• Prompts for phase 2 (pp. 20–23) |

| Oral and mental starter | Main teaching | Notes | Plenary |
| --- | --- | --- | --- |
| Use whiteboards; keep a good pace.<br>These fractions or ratios simplify to $\frac{2}{5}$, $\frac{3}{4}$, $\frac{4}{5}$ or $\frac{2}{3}$. Which is which?<br>• $\frac{24}{36}$, $\frac{36}{90}$, $\frac{36}{48}$<br>• 28 : 70, 45 : 60, 36 : 45<br><br>Give me two numbers:<br>• first is 25% of second;<br>• second is $2\frac{1}{2}$ times first;<br>• second is 150% of first;<br>• ratio of first to second is 2 : 5.<br><br>Defining percentages as hundredths and including examples >100%:<br>• use 1% = $\frac{1}{100}$ to convert percentages to fractions;<br>• use 1 = 100% to convert fractions to percentages, e.g. $\frac{3}{4} = \frac{3}{4}$ of 100%;<br>• use 1% = 0.01 to convert percentages to decimals and vice versa. | **Multiplicative relationships**<br><br>**Stage 1:** Using linking blocks pupils construct two strips, each a different colour. Model the process of expressing:<br><br>• no. of black blocks : no. of white blocks = 2 : 5,<br>• no. of white blocks : no. of black blocks = 5 : 2;<br>• no. of black = $\frac{2}{5}$ × no. of white, no. of white = $\frac{5}{2}$ × no. of black;<br>• no. of black = 0.4 × no. of white, no. of white = 2.5 × no. of black;<br>• no. of black = 40% of no. of white, no. of white = 250% of no. of black.<br><br>**Stage 2:** Pupils work in pairs, using linking blocks to complete a similar set of expressions for the ratio 3 : 4, perhaps recording on resource sheet FR6. Then discuss this in a mini-plenary.<br>**Stage 3:** Give a set of problems on cards, all based around the ratios 2 : 5 and 3 : 4 and their inverses (see resource sheet FR9 for a suggested set). In groups of four, pupils sort the problems into two sets, according to the ratio with which they are associated. In pairs, the pupils then solve the problems. Finally, each pair checks one or two answers of the other pair.<br><br>In the following lesson, consolidate and extend by repeating the above for a different pair of ratios, e.g. 4 : 5 and 2 : 3 (resource sheets FR7 and FR8). The introductory modelling can be kept brief, with more time devoted to the plenary. (See resource sheet FR10 for a set of problems.)<br><br>**Mixed problems**<br>Give a set of mixed problems for pupils to solve in pairs (see resource sheet FR11). The problems are the same types as before, but related to a mixture of ratios or fractions.<br>Plan a mini-plenary to discuss solutions as before (see first note in plenary column), allowing time for the final plenary.<br>Further contexts: Framework pp. 60, 66, 68, 72, 74, 78, 80, 98. | The process could be modelled using the ITP *Ratio strips* or an acetate version of resource sheet FR5 (also used by pupils for recording). Exploring the relationship both ways encourages flexible thinking and sows the seeds of ideas of inverses to be developed later.<br><br>**Support:** If pupils are having difficulty solving problems, intervene to discuss one or two examples. In classes of mixed attainment, group pupils and distribute tasks so that all are able to contribute.<br><br>**Extension:** Ask pupils to compose their own sets of problems related to a specific ratio. | Select two or three problems to discuss:<br>• how pupils identified links to the chosen ratio;<br>• whether they found it useful to represent the problem with blocks or diagrams;<br>• alternative methods of solution, e.g. converting a percentage to a fraction or decimal calculation;<br>• whether the answer makes sense in terms of order of magnitude, units, etc.<br><br>**Final plenary**<br>Return to the OHT or poster of ideas from the first lesson. Ask pupils:<br>• to add new entries, links or diagrams;<br>• what new ideas, connections and applications have helped them make most progress;<br>• what they need to do more of next time you visit the topic.<br>(See prompts p. 23.) |

# Supplementary notes

## Key lesson: Fraction operators

### Introduction

Number 2 is the first unit in Year 7 that requires pupils to draw on their prior knowledge of fractions, decimals, percentages, ratio and proportion. The first purpose of this lesson is to provide an opportunity for pupils to show what they know, understand and can do. The benefits to be gained from this are:

- pupils are given an opportunity to refresh their memories, to learn from each other and to have their learning acknowledged;
- the teacher can assess pupils' knowledge in an informal and informative way;
- the teacher can better judge the pitch of subsequent work and make links with previous learning.

The following plan models one possible way of structuring the first lesson to encompass these aims, keeping the timings flexible:

- The starter is extended, to perhaps 20 minutes. This is in order to engage the whole class in a review of their previous work on fractions, decimals and percentages. It begins with active group work, getting pupils to contribute from their prior knowledge.
- The main activity (and the main activity of the next lesson) develops images of the meaning of fractions as numbers and as operators. The review of earlier ideas and connections are revisited and refined through the use of discussion and captured on web diagrams. Pupils are encouraged to recognise the new challenge of using fractions as operators and to link this to their earlier understanding.
- The plenary refocuses attention on these connections by getting pupils to articulate them.

# Key lesson plan: Fraction operators

## Objectives

- Use fraction notation to describe parts of shapes and to express a smaller whole number as a fraction of a larger one; **identify equivalent fractions.**
- Calculate simple fractions of quantities (whole-number answers); multiply a fraction by an integer.
- Consolidate and **extend mental methods of calculation to include decimals, fractions and percentages**, accompanied where appropriate by suitable jottings.
- Represent problems mathematically, making correct use of symbols, words and diagrams.

## Starter (extended)

**Vocabulary**
fraction
decimal
percentage
equivalent
equal
of

**Resources**
mini-whiteboards
large sheets of plain paper
OHP or flipchart

**Reviewing Key Stage 2 work**

Display the six key words in the vocabulary list.

**Q** Which key words do you recognise from Year 6?

**Q** Can you give me a fact or expression related to one or more of the key words?

For example, $\frac{5}{10}$ is equivalent to $\frac{1}{2}$, $\frac{1}{10}$ is the same as 10%.

Take responses from pupils, perhaps using whiteboards. Encourage them to use words, numbers, symbols and pictures. Discuss some of their examples.

**Q** Can you give another example, to include a statement with the equals sign?

**Q** Can you give an example with a labelled diagram?

**Q** Can you use the same example for a different key word?

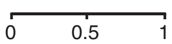
Ask pupils in pairs to record on blank paper particular things they know, related to the familiar words. Encourage a variety of expressions: equivalences, labelled diagrams, number lines, statements in words, etc. For example:

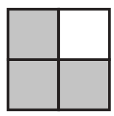
$\frac{1}{2} = \frac{2}{4}$

$\frac{1}{3}$ of 12 = 4

$\frac{1}{4}$ = 25%

$\frac{1}{2}$ = 0.5



$\frac{3}{4}$

Circulate during this work, to stimulate engagement with the task and note examples which will be useful to share.

Note: Keep pupils' contributions on OHTs or posters, to display again in the final plenary of the unit.

Next, invite pupils to the front of the class to write up an example chosen from their collection, preferably on an OHT or flipchart. Encourage a variety of responses, perhaps inviting lower attaining pupils first:

**Q** Is there another way to write this?

**Q** Is there another diagram to show this?

Spend a few minutes discussing the collected examples, making links between them where appropriate. Say that you are saving the contributions for later. Conclude by explaining that in this topic pupils will be linking their ideas together, meeting some new ideas and using fractions in practical contexts.

## Main activity

**Vocabulary**
operator (‘of’)
improper fraction
mixed number
increase
decrease

**Resources**
*Either:*
Two copies each of OHTs ‘Seven stack’ and ‘Seven stack in thirds’ (resource FR1)
*or:*

ITP *Images of fractions*

*Also:*

Further copies of OHT ‘Seven stack in thirds’ (FR1), shaded to show $5 \times \frac{1}{3}$ and $\frac{1}{3}$ of 5

*If repeating for quarters:*

Two further copies of OHT ‘Seven stack’ (FR1) and two copies of OHT ‘Seven stack in quarters’ (FR2)

**Making connections**

Lead into the next activity by saying that you will show some diagrams to help pupils illustrate facts they have noted. The diagrams will also link ideas together and help pupils to see why certain facts are true or certain calculations equivalent. If possible, start with an example from pupils’ contributions, finding a fraction of an amount, such as $\frac{1}{3}$ of 15 = 5.

**Q** Can you give me more examples of finding ‘$\frac{1}{3}$ of’?

Record the examples which the class give.

**Q** Can you tell me what we are doing to the number 15 when we find one third of it?

**Q** Which operation is this?

Draw from pupils that the amount is divided into 3 equal parts and that this is what happens both when ‘dividing by 3’ and when finding ‘a third of’.

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*Now follow pages 14–16 of the stage 1 prompts for main activities in phase 1: ‘Images of thirds: making the connection between seven thirds and one third of seven’.*

*The prompts develop images of thirds, using the ITP ‘Images of fractions’, OHTs from the resource bank, or drawings on the board or a flipchart:*

- *multiples of one third, e.g.* $7 \times \frac{1}{3}$;
- *thirds of numbers, e.g.* $\frac{1}{3}$ *of 7.*

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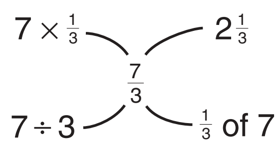
Now tell pupils that you are going to pool ideas from the start of the lesson, together with new links which the diagrams may have helped them to make. Illustrate these links in a web diagram of expressions equivalent to $\frac{7}{3}$:




*(You could add $\frac{1}{3} \times 7$, but the explanation of this is suggested for a later stage – see the prompts on page 17.)*

Draw attention to the connections illustrated in the diagram. Trace your finger along the links and ask pupils to find ways of summarising them.

**Q** What is the link between ... and ...?

**Q** How do we know?

**Q** Does anyone think of this in a different way?

**Q** Can anyone add to that explanation?

Note particularly that you have looked at a diagram which helps you to see that seven thirds is the same as seven divided by three.

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Show shaded diagrams for $5 \times \frac{1}{3}$ and $\frac{1}{3}$ of 5. Ask pupils to work in pairs to draw a web diagram for $\frac{5}{3}$.

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Depending on the length of the lesson you could now:

- **either** repeat the above sequence for quarters (perhaps leaving fifths until the next lesson);
- **or** move on to the plenary.

 

## Plenary

Copy one of the web diagrams produced. For example:

$$\begin{array}{ccc} 7 \times \frac{1}{3} & & 2\frac{1}{3} \\ & \frac{7}{3} & \\ 7 \div 3 & & \frac{1}{3} \text{ of } 7 \end{array}$$

Rub out all entries except $2\frac{1}{3}$.

**Q** What difference would it make to the other entries if this was $2\frac{2}{3}$?

Invite pupils to complete the diagram.

**Q** What if the entry was simply 2?

Invite pupils to complete the diagram. The following is likely to appear:

$$\begin{array}{ccc} 6 \times \frac{1}{3} & & 2 \\ & \frac{6}{3} & \\ 6 \div 3 & & \frac{1}{3} \text{ of } 6 \end{array}$$

**Q** Is this the only web which would have an entry of 2?

Invite other suggestions – you may need to suggest a starting point such as $\frac{8}{4}$ in the centre.

Encourage pupils to explain links as other examples are considered.

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Remind pupils where the lesson started and how much they have revealed of what they know. Point to one of the web diagrams.

**Q** Is there any connection in the diagram which you understand in a new way?

Explain that over the next few lessons you will discuss some more images and links and use these to solve problems.

### Remember

- Fractions can be less than 1, equal to 1 or greater than 1.
- We can think of a fraction such as $\frac{7}{3}$ (write on board) as seven lots of one third or as one third of seven. Seven thirds means the same as seven divided by three.
- In Key Stage 2 fractions were often drawn as points on a number line or parts of a shape (e.g. a pizza). In Key Stage 3 we add to these images by using the idea of a stack (e.g. the seven stack) to demonstrate using fractions in multiplication.

# Prompts for phase 1

## Fraction operators

### Rationale

The first of the following sequences (which are much easier to demonstrate than to explain on paper) establishes connections between multiples of a fraction and fractions of a quantity, leading to a recognition of the equivalence of different expressions, such as $\frac{7}{3}$, $7 \times \frac{1}{3}$, $\frac{1}{3}$ of 7 and $7 \div 3$. In the second sequence, this is extended to finding non-unitary fractions of a quantity, for example $\frac{5}{3}$ of 7.

Note that:

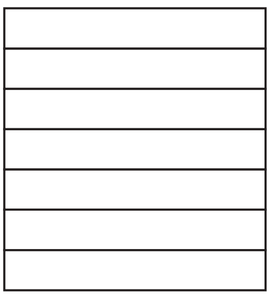
- Pupils will have met the representation of a fraction as part of a single shape. What may be new in the image here is stacking a set of rectangular strips, each representing 1, to represent a bigger number. Descriptions assume use of an OHP, alternatives being the interactive teaching program (ITP) *Images of fractions* or hand drawings on a flipchart or board. The inaccuracy of hand drawn lines should be discussed but should not be a problem.
- Drawing on their understanding of symbols and language, pupils will usually read '×' as 'times', meaning 'lots of'. When dealing with fractions it makes sense to interpret 'of' as meaning 'times' and to write '×' in place of the word. (Eventually pupils realise the order of numbers in multiplication is unimportant.)

The development of the main part of the first two lessons in phase 1 is now described in two stages. The description uses thirds. Quarters and fifths can be approached in a similar way. Refer to the unit plan for a suggested teaching order.

### Stage 1

**Images of thirds: making the connection between seven thirds and one third of seven**

When working through this sequence it is important to emphasise that the stack of strips represents the number seven.



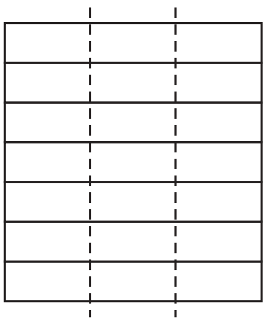
Show OHT 'Seven stack' (from resource FR1), covering all but the top strip. Say:

Here is a strip representing the number 1.

Now reveal the strips one by one. Say:

Together, count the strips as I show them: 1, 2, 3, ..., 7.

The whole stack represents the number 7.



**Q** How could I draw vertical lines to divide the stack of seven into three equal parts?

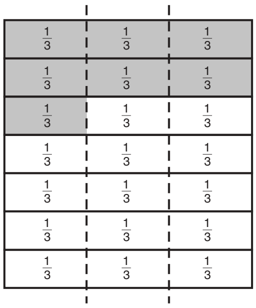
The lines do not have to be exactly right – we can imagine that they are!

**Q** How could we check that each part is the same?

**Q** If I look at any one strip how could we label each small section?

**Q** Can I mark every small section as $\frac{1}{3}$? Why?




Show OHT 'Seven stack in thirds' and say:

Here is a marked-up diagram.

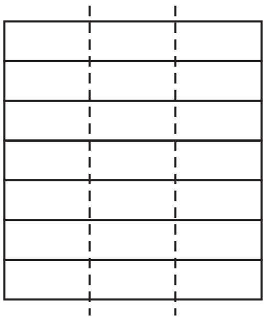
**Q** How many thirds are there altogether? **(record $\frac{21}{3}$ = 7)**

We will now count the first few multiples of one third together and record them.

Beginning at the top left, count from left to right along the strips, shading each $\frac{1}{3}$ as you go.

| Say together: | Record: |
|---|---|
| one times a third is one third | $1 \times \frac{1}{3} = \frac{1}{3}$ |
| two times a third is two thirds | $2 \times \frac{1}{3} = \frac{2}{3}$ |
| three times a third is three thirds which is one | $3 \times \frac{1}{3} = \frac{3}{3} = 1$ |
| four times a third is four thirds which is one and a third | $4 \times \frac{1}{3} = \frac{4}{3} = 1\frac{1}{3}$ |
| ... | ... |
| **seven times a third is seven thirds which is two and a third** | $\mathbf{7 \times \frac{1}{3} = \frac{7}{3} = 2\frac{1}{3}}$ |

Say that you will come back to this diagram.



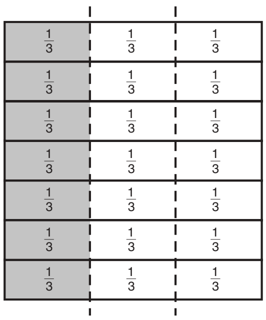
Now start with a fresh copy of OHT 'Seven stack'. As before, cover all but the top strip, then reveal them one by one.

Together, count the strips as I show them: 1, 2, 3, ..., 7.

As before, the whole stack represents the number 7.

**Q** In the last example, how did we divide the stack into three equal parts?

**Q** Looking at any one strip, how did we label each small section?




Show OHT 'Seven stack in thirds' (from resource FR1) and say:

Here is a marked-up diagram.

**Q** When we divide any shape into three equal parts how could we describe what we have done to the shape?

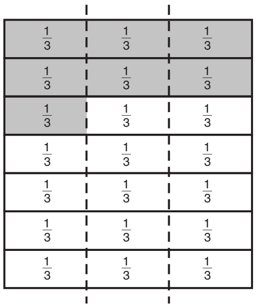
**Q** Could someone shade one third of this whole stack?

**Q** If I cover some of the strips is it still correct to say that one third of the visible stack is shaded?

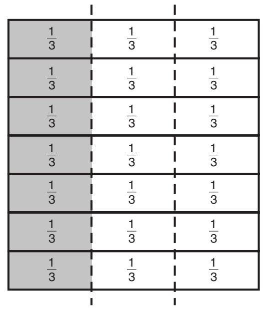
Cover the lower part of the stack, revealing a strip at a time as you count from top to bottom.

| Say together: | Record: |
|---|---|
| one divided by three is one third of one which is one third | $1 \div 3 = \frac{1}{3}$ of $1 = \frac{1}{3}$ |
| two divided by three is one third of two which is two thirds | $2 \div 3 = \frac{1}{3}$ of $2 = \frac{2}{3}$ |
| ... | ... |
| **seven divided by three is one third of seven which is seven thirds** | $\mathbf{7 \div 3 = \frac{1}{3}}$ **of** $\mathbf{7 = \frac{7}{3}}$ |

**Q** Imagine the stack is larger. Could we continue counting?



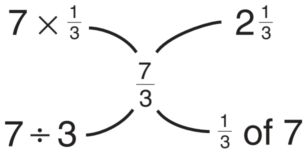
Now display the two diagrams side by side.



**Q** Do you remember how we counted seven thirds in the first diagram?

**Q** Do you remember how we counted to find one third of seven in the second diagram?

**Q** Are the answers the same? Why? Can we convince everyone?




Point out that you have linked lots of equivalent expressions. Illustrate these links in a web diagram, as on the left.

Point to different expressions in the web and ask pupils to read and relate each expression to one of the diagrams. *(See the notes for the key lesson on page 12.)*

**Q** Could we draw a similar web diagram for five thirds? For two thirds? ...

### Repeating stage 1 for quarters and fifths

A decision needs to be made depending on the extent to which pupils appreciate the generality of the image, i.e. they can apply it to other fractions and see a definition of a non-unit fraction which is different to collecting together a number of unit fractions. For many classes, a repetition of the sequence for quarters and fifths will be appropriate. As a test of their understanding you might ask:

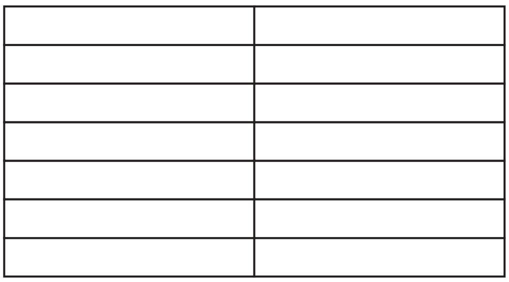
**Q** Can you draw a web of expressions equivalent to $\frac{11}{8}$ and explain all the connections in your diagram?

## Stage 2

### Images of thirds: extending beyond unit fractions to thirds of seven

In this second sequence, as well as extending beyond unit fractions of quantities, part of the aim is to create a link between ‘lots of’ and ‘fractions of’, hence linking the term ‘of’ with the operation multiplication.

Show OHT ‘Two seven stacks’ (from resource FR3) and say:



**Q** Can you see how this diagram shows two lots of seven?

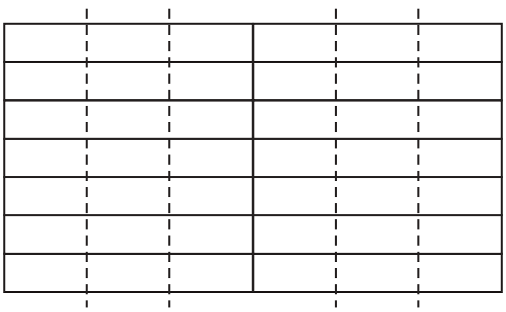
Now cover the right-hand stack.

**Q** How could I divide this lot of 7 into three equal parts?

Now cover the left-hand stack.

**Q** How could I divide this lot of 7 into three equal parts?

Show OHT ‘Each seven stack in thirds’ (resource FR3), saying: ‘Here is a marked-up diagram.’



Cover the right-hand part of the stack, revealing a single column at a time as you count from left to right.

Say together: | Record:

| | |
|---|---|
| one third of seven is seven thirds | $\frac{1}{3}$ of 7 = $\frac{7}{3}$ |
| two thirds of seven is fourteen thirds | $\frac{2}{3}$ of 7 = $\frac{14}{3}$ |
| **three thirds of seven is twenty-one thirds...** | $\mathbf{\frac{3}{3}}$ **of 7 =** $\mathbf{\frac{21}{3}}$ **= 7** |
| *(point out this is the same as the initial stack, one lot of 7)* | |
| four thirds of seven is twenty-eight thirds | $\frac{4}{3}$ of 7 = $\frac{28}{3}$ |
| five thirds of seven is thirty-five thirds | $\frac{5}{3}$ of 7 = $\frac{35}{3}$ |
| **six thirds of seven is forty-two thirds** | $\mathbf{\frac{6}{3}}$ **of 7 =** $\mathbf{\frac{42}{3}}$ **= 14** |

Reflect on the final picture which shows 2 'lots of' 7.

Say together: | Record:

| | |
|---|---|
| two **lots of** seven is two **times** seven | 2 'lots of' 7 = 2 × 7 |
| six thirds **of** seven is six thirds **lots of** seven | $\frac{6}{3}$ of 7 = $\frac{6}{3}$ 'lots of' 7 |
| which is six thirds **times** seven | = $\frac{6}{3} \times 7$ |

Go back through the other 'fractions of' 7:

- Say and record in terms of multiplication.
- Ask pupils whether the answer will be less than, equal to or greater than 7.

Record these facts, asking pupils to provide the explanations:

- $\frac{1}{3}$ of 7 is the same as $\frac{1}{3} \times 7$ – 'of' means 'multiply'.
- These are the effects of fraction multipliers on the number 7:
  - $\frac{1}{3} \times 7$ and $\frac{2}{3} \times 7$ have answers less than 7 (a decrease).
  - $\frac{3}{3} \times 7$ is equal to 7.
  - $\frac{4}{3} \times 7$, $\frac{5}{3} \times 7$ and $\frac{6}{3} \times 7$ have answers more than 7 (an increase).
- The answer to $\frac{5}{3} \times 7$ can be found by doing the calculation $(7 \div 3) \times 5$.

**Q** If I want to find '$\frac{5}{3}$ of' any number, will it be an **increase** in the number or a **decrease**? Why?

**Q** Using a calculator how would you find $\frac{2}{3}$ of 7? Or $\frac{4}{3}$ of 7? ...

**Q** Could we do something similar for other fractions such as quarters or fifths?

# Ratios and equivalence

## Rationale

Fractions are first encountered as parts of a whole. At the beginning of Key Stage 3 a broader view is needed because many applications use fractions as a way of comparing one quantity with another. Linking blocks or coloured rods provide a good image, adding a kinaesthetic component to help pupils grasp the relationships. Drawings still have a place, such as when using the OHP or individual pupil whiteboards. The terminology of ‘strips’ and ‘blocks’ was chosen to cover both forms of representation.

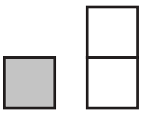
Pupils should have knowledge of equivalent fractions from Key Stage 2. However, the approach of building up the number of blocks in each strip establishes equivalences in a way which may be new to them. They will also have some experience of ratio and proportion but may not have met ratio notation before. The strips of blocks provide an effective image for linking ratio and fraction notation.

Pupils need to understand the links between different notations. The equivalence between $\frac{1}{2}$ and $1 \div 2$ was discussed in the first two lessons and the link between $1 : 2$ and $\frac{1}{2}$ will be discussed in this lesson. There is an essential equivalence between the expressions, in that they describe the same relationship in different ways: $1 : 2$ stresses the *parts* involved, $1 \div 2$ stresses the *operation* of division and $\frac{1}{2}$ stresses the *result* of the operation.

The main teaching activity and the plenary of this lesson are now described.

## Main activity

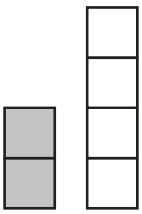
**Introducing ratio notation and establishing equivalences**



Ask pupils to set up the arrangement on the left, using linking blocks or similar.

We can say that ‘the number of black blocks is $\frac{1}{2}$ the number of white blocks’. Another way of expressing this is ‘the *ratio* of the number of black blocks to the number of white is 1 to 2’, noting that the ratio ‘1 to 2’ is usually written as ‘1 : 2’.

Now ask pupils to double up the number of blocks in each strip, explaining that there is one black block for every two white.

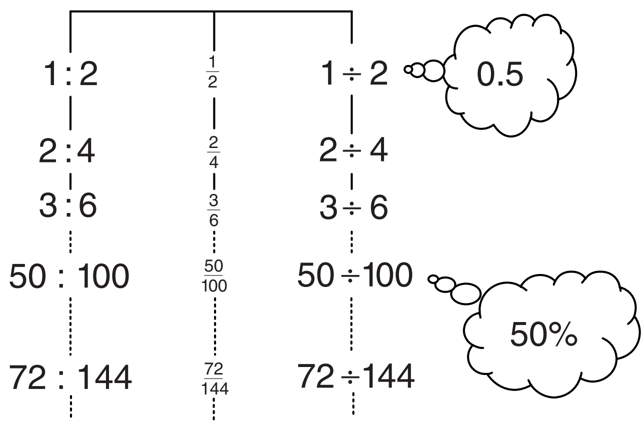


We can say that ‘the number of black blocks is now $\frac{2}{4}$ of the number of white’, or ‘the ratio of the number of black blocks to the number of white is now 2 to 4’. But the number of black is still half the number of white:

‘$\frac{2}{4}$ is equivalent to $\frac{1}{2}$’ and ‘2 : 4 is equivalent to 1 : 2’.

Extend the sequence of equivalent fractions and ratios by adding more blocks to each strip.

Incorporating the link with division from lessons 1 and 2 as well as ratio from this lesson, construct a diagram or table based on $\frac{1}{2}$. Encourage pupils to extend the diagram beyond the equivalences represented with blocks and to add decimals and percentages.



The unit plan suggests repeating the above for other unit fractions, such as $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$. For each, pupils build up a diagram of equivalent ratios, divisions and fractions.

## Plenary

Show strips of an unsimplified ratio, such as 6 : 18.

**Q** What fraction is this equivalent to and why?

**Q** Could I see this as $\frac{1}{3}$?

**Q** Could I see this as $\frac{3}{9}$?

Encourage pupils to express ways of 'seeing' the result, rather than talking in terms of 'cancelling' numbers:

- To see $\frac{1}{3}$ pupils may say that three black strips would make the white strip. Or they may say that there is one black block for every three white blocks.
- To explain $\frac{3}{9}$ pupils will need to see the black made up of 3 lots of two and the white made up of 9 lots of two – or 3 black blocks for every 9 white blocks. (Strips suitably coloured may be useful to illustrate.)

# Prompts for phase 2

## Multiplicative relationships

### Rationale

Phase 1 explored fractions as operators and established ratio notation. Phase 2 explores multiplicative relationships within a restricted set of simple fractions/ratios (2 : 5 and 3 : 4 then later 4 : 5 and 2 : 3), expressing operators in fraction, decimal and percentage forms. Following the principle of studying operations and their inverses together sows the seeds of a more formal study later on:

- it encourages flexible thinking, since either of the two quantities involved can be thought of as the unit against which the other is compared;
- it ensures working with operators which are less than 1 and operators which are greater than 1.

Keeping to simple ratios facilitates understanding of the relationships to be expressed. The same principle applies to the sets of problems which follow:

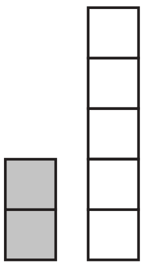
- Pupils can draw on their awareness of the relationships, equivalences and notations they have encountered.
- Their methods may be informal, but they start to think multiplicatively.
- A way of representing the problem is available to them if they wish, or if the teacher thinks it may help them when stuck.

The development of the main part of the first lesson in phase 2 is now described in three stages.

### Stage 1

**Exploring the ratio 2 : 5**

Model the recording for pupils on the board, on the OHP (OHT of resource FR5) or using the ITP *Ratio strips*. Pupils can quickly make their own paper record on a copy of resource sheet FR5.



Ask pupils to use linking blocks of two chosen colours to construct the strips shown on the left.

First express the relationship between the number of blocks in each strip (both ways round) using the language of ratio:

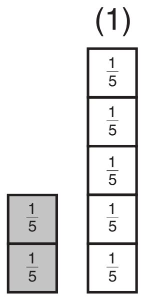
No. of black blocks : no. of white blocks = 2 : 5

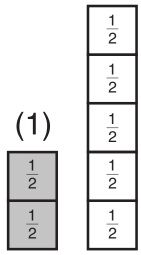
No. of white : no. of black = 5 : 2

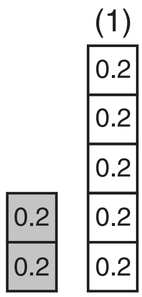
Now, taking each strip in turn as the ‘unit’ (worth ‘one’), express the relationship between the number of blocks, considering fraction, decimal and percentage equivalents.

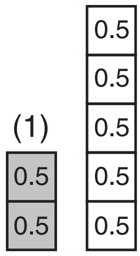
- Clarify which strip is being taken as the unit (note: 100% = 1).
- Express each block as a fractional part of the unit.
- Express the second strip as a fraction of the chosen strip.

| | |
|---|---|
| **Think of the *white* strip as unit (fraction).**<br>The white strip is worth ‘one’.<br>**Q** What is each white block worth (as a fraction)?<br>(Mark each white block as $\frac{1}{5}$.)<br>Establish each black block is worth $\frac{1}{5}$.<br>(Mark each black block $\frac{1}{5}$.)<br>**Q** What fraction is the number of black blocks of the number of white blocks?<br>Record:<br>**No. of black blocks = $\frac{2}{5}$ × no. of white blocks** | **Think of the *black* strip as unit (fraction).**<br>The black strip is worth ‘one’.<br>**Q** What is each black block worth (as a fraction)?<br>(Mark each black block as $\frac{1}{2}$.)<br>Establish each white block is worth $\frac{1}{2}$.<br>(Mark each white block $\frac{1}{2}$.)<br>**Q** What fraction is the number of white blocks of the number of black blocks?<br>Record:<br>**No. of white blocks = $\frac{5}{2}$ × no. of black blocks** |
| **Think of the *white* strip as unit (decimal).**<br>The white strip is worth ‘one’.<br>**Q** What is each white block worth (as a decimal)?<br>(Mark each white block as 0.2.)<br>Establish each black block is worth 0.2.<br>(Mark each black block 0.2.)<br>**Q** What decimal fraction is the number of black blocks of the number of white blocks?<br>Record:<br>**No. of black blocks = 0.4 × no. of white blocks** | **Think of the *black* strip as unit (decimal).**<br>The black strip is worth ‘one’.<br>**Q** What is each black block worth (as a decimal)?<br>(Mark each black block as 0.5.)<br>Establish each white block is worth 0.5.<br>(Mark each white block 0.5.)<br>**Q** What fraction is the number of white blocks of the number of black blocks?<br>Record:<br>**No. of white blocks = 2.5 × no. of black blocks** |
| **Think of the *white* strip as unit (percentage).**<br>The white strip is worth 100%.<br>**Q** What is each white block worth?<br>(Mark each white block as 20%.)<br>Establish each black block is worth 20%.<br>(Mark each black block 20%.)<br>**Q** What percentage is the number of black blocks of the number of white blocks?<br>Record:<br>**No. of black blocks = 40% × no. of white blocks** | **Think of the *black* strip as unit (percentage).**<br>The black strip is worth 100%.<br>**Q** What is each black block worth?<br>(Mark each black block as 50%.)<br>Establish each white block is worth 50%.<br>(Mark each white block 50%.)<br>**Q** What percentage is the number of white blocks of the number of black blocks?<br>Record:<br>**No. of white blocks = 250% × no. of black blocks** |

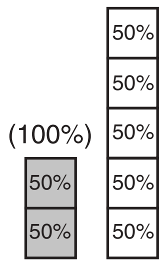






Points to consider:

- Pupils may want to write $\frac{5}{2}$ as $2\frac{1}{2}$, but do not lose the improper fraction form as a way of expressing the relationship.
- Perhaps introduce the vocabulary of *inverse* operators when talking about the relationships expressed.

## Stage 2

**Exploring the ratio 3 : 4**

Ask the class to work in pairs to complete a similar set of eight expressions for the ratio 3 : 4, using resource sheet FR6 for recording. Follow this with a mini-plenary to check pupils' work and deal with any common difficulties (e.g. it may be necessary to clarify that $\frac{4}{3} = 133\frac{1}{3}\%$).

## Stage 3

**Solving problems related to the ratios 2 : 5 and 3 : 4**

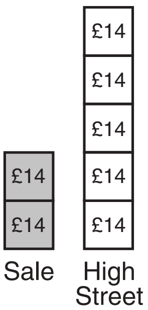
Present a set of problems, all based around the ratios 2 : 5 and 3 : 4 and their inverses. (Resource sheet FR9 gives a varied set of problems which you could adapt to suit your own classes.) Prepare the problems on separate cards or slips of paper, one set per four pupils.

- In groups of four, pupils divide the problems into two sets: those associated with 2 : 5 and those associated with 3 : 4. In some cases, the classification may not be obvious but pupils should be able to sort the problems using informal strategies based on relative sizes.
- Each pair then takes one of the two subsets to solve.
- Pairs check each other's work and discuss problems where the classification was uncertain or incorrect.

It is intended that pupils should use informal methods for solving these problems, drawing on understanding gained from working with the strips, but using the strips for explicit representation of a problem only when it seems helpful. For this reason, it is recommended to let pupils tackle the problems without further guidance, intervening only when they encounter difficulties that they cannot overcome by talking within their pairs or fours.

Here are two examples of how pupils might be helped by using linking blocks and/or sketch diagrams to represent the situation and find, or perhaps just confirm, the answer.

> *An alternative clothing sale offers jeans at £28. These are £70 in high-street shops. Is this less than half price? How could you use fractions, ratios or percentages to compare the alternative price to the high-street price?*



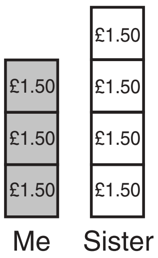

Initial classification of this problem may provoke some discussion. Some pupils might recognise 28 : 70 as equivalent to 2 : 5 and classify accordingly. Or they might argue that it is less than half price and therefore must be related to 2:5 rather than 3:4.

The sale price divides into two parts: $\frac{1}{2} \times £28 = £14$

The high street price divides into five parts: $\frac{1}{5} \times £70 = £14$
(or confirm $5 \times £14 = £70$).

The amounts fit with the ratio 2 : 5. So the sale price is 40% of high-street price.




*My sister lets me help on her paper round. It pays £10.50 and she shares this between me and her in the ratio 3 : 4. (a) How much do I get? (b) How much does my sister get?*

The diagram shows the £10.50 must be divided into seven parts, each part being $\frac{1}{7} \times$ £10.50 = £1.50.

(a) I get three parts: $\frac{3}{7} \times$ £10.50 = £4.50

(b) My sister gets four parts: $\frac{4}{7} \times$ £10.50 = £6

For the plenary of this lesson, please see the unit plan.

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**Repeating the lesson with a different data set**

Repetition helps to consolidate learning and strengthen pupils' understanding of links. The pattern of the previous lesson can be repeated by changing only the data set:

- choose a different pair of ratios within the set of small numbers 2, 3, 4 and 5, such as 4 : 5 and 2 : 3;
- expect pupils to take a more active role from the beginning, with less need for modelling by the teacher.

Resource sheets FR7 and FR8 are suitable for recording relationships for the two ratios and sheet FR10 gives a set of problems related to these ratios. As before, it is worth cutting the problems up separately, for pupils to sort.

---

**Mixed problems and final plenary**

The suggested way of concluding the unit is to give a set of mixed problems for pupils to solve in pairs. Resource sheet FR11 gives a possible set, to be adapted as needed. It includes similar problems to before, but related to mixed ratios/fractions.

For this lesson, plan a mini-plenary to discuss solutions, allowing time for the final full plenary, which refers pupils back to the key lesson and reviews progress made.

In the final plenary, return to the OHTs or poster of ideas from the first lesson. Ask these questions, allowing a short discussion in pairs at each stage:

**Q** Can we make any additional entries or new links? *(use key words)*

**Q** What images helped to explain the connections? *(add sketches to the poster)*

We have looked at:

- new ideas and ways of recording these ideas;
- new connections between ideas;
- applying these ideas to solving problems.

**Q** Which of these has helped you make most progress?

**Q** What do you need to do more of next time we visit the topic?

# Where next?

## Year 7 units including work on FDPRP

For schools that are following the sample medium-term plans, this unit stands in place of Number 2, but also incorporates a main objective from Number 4. Number 4 could therefore be planned to address similar objectives to Number 2, building in application of the skills, thus addressing the more challenging problem-solving objectives.

To consolidate and extend the work you could for example:

- repeat the starters of phase 2, to give pupils more practice and help them automate the processes involved;
- deepen pupils' awareness of interconnections between multiplication, division, fractions, ratio and proportion, emphasising fractions as operators;
- develop problem-solving strategies with more varied and challenging problems, but still keeping to informal methods for dealing with ratio and proportion.

This approach helps pupils to make and reinforce links between mathematical concepts and notation. Dealing with the concepts separately has been found to make it much more difficult for pupils to see these connections.

To give an illustration of how to consolidate and extend the work, consider the images used to illustrate multiples of a fraction and fractions of a number. From these a series of oral and mental starters could evolve, eventually working just orally to gain fluency. For example, chant these fraction tables as a class (answers can be given as improper fractions or mixed numbers):

- one times a third is a third; two times a third is two thirds; three times a third is three thirds which is 1; ...
- one third of one is a third; one third of two is two thirds; ...

Extend perhaps to non-unitary fractions:

- one third of seven is seven thirds ($2\frac{1}{3}$); two thirds of seven is fourteen thirds ($4\frac{2}{3}$); three thirds of seven is twenty-one thirds (7); ...
- one quarter of seven is seven quarters; two quarters of seven are fourteen quarters; three quarters of seven are twenty-one quarters; ...
- one fifth of seven is seven fifths; two fifths of seven is fourteen fifths; ...

The other Year 7 unit to include an element of FDPRP is Number 5. Here you could:

- place work on addition and subtraction of fractions displaced from the original Number 2, using pupils' greater knowledge of equivalent fractions;
- include the two lessons available as plans to support Year 7 to Year 8 transition (www.standards.dfes.gov.uk/keystage3/publications).

The website materials offer a way of reviewing progress in one key area at the end of the year, and helping transition to the following year. The website also includes suggestions for transition from Year 8 to Year 9 within this same strand of mathematics. The materials have something to offer all schools, but may be particularly valuable in circumstances where pupils transfer from a middle school to an upper school.

### Year 8 and Year 9 units on proportional reasoning

The suggested sequence would provide a basis for moving on to other mini-packs (Year 8 multiplicative relationships and Year 9 proportional reasoning), developing the ideas further to more formal methods for dealing with general numbers. Common threads through all these units include:

- the focus on fractions as multiplicative operators;
- the emphasis on equivalent forms of fractions, ratios, decimals and percentages;
- dealing with inverse operations from the earliest stages;
- using line segments (at first accurately scaled, later as sketches) as an image for fractions, ratio and scaling;
- developing proportional thinking and methods for solving proportion problems.

As pupils move through the key stage you should find that they are better prepared for these later units, so that you can increase your expectations of the standards they should achieve.