Guidance

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for school improvement

Teaching mental mathematics from level 5: measures and mensuration in algebra

Mathematics teachers and consultants

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Contents

Teaching mental mathematics from level 5: measures and mensuration in algebra

Introduction

This appendix is one of four addressing measures and mensuration within other strands of mathematics. Each appendix is offered as a supplement to the associated mental mathematics booklet and should be read in conjunction with it. Each one makes use of the principles and activities described in the corresponding booklet. The appendices are:

- *Measures and mensuration in number*;
- *Measures and mensuration in algebra*;
- *Measures and mensuration in shape and space*;
- *Measures and mensuration in handling data* (*scheduled later*).

Measures and mensuration are developed as part of the other strands in order to illustrate ways of incorporating these important aspects of mathematical thinking into a variety of teaching opportunities. It is through identifying the interconnections between different concepts and contexts in mathematics that pupils are better able to confirm, use and apply their existing knowledge and skills.

The activities presented here are intended to engage teachers and pupils in collaborative work with an emphasis on dialogue. Through discussion, pupils have a greater chance of understanding the 'big picture' as they begin to reprocess thinking and make new connections.

Measures and mensuration in the context of algebra

The topics covered in this supplement are selected from the **Measures and mensuration** section and the **Algebra** (equations, formulae and identities) section of the yearly teaching programmes in the *Framework for teaching mathematics: Years 7, 8 and 9.*

The activities described in this supplement adapt and develop activities suggested in *Teaching mental mathematics from level 5: algebra* (DfES 1287-2005) and are designed to support pupils in developing their sense of variables and applying their knowledge and understanding of algebraic conventions and solving equations in the context of functions and formulae. The activities focus on engaging pupils with the application of algebra through discussion and collaboration and on providing opportunities to refine their mental images. The guidance and charts on the following pages can be used to help adjust the challenge of tasks in order to keep pupils on the edge of their thinking.

Tackling problems involving measures and mensuration can help pupils to make connections between different aspects of mathematics. For example, formulae are derived to record relationships and enable us to perform calculations for all shapes that share a given set of properties. For example, areas of circles make use of a simple formula linking area and radius (two variables). When we consider mapping values of the area to the radius we are dealing with a quadratic function that has a real application for part of its range. In this sense the context of shape and space (through mensuration) offers pupils an opportunity to apply an algebraic generalisation.

It is particularly important when pupils are trying to model a real-life context through the use of mathematics that they are supported in making connections in their understanding of different forms of representation and that they are able to choose appropriate forms to help solve a problem. The activities that follow provide opportunities for pupils to refine their mental images by discussing and describing these different forms of diagrammatic, algebraic, numerical and graphical representation.

Measures

Estimation

Working with measures can provide an ideal context in which to develop pupils' skills in using and applying new knowledge from different strands of mathematics. For example, as pupils make decisions about the measures in a problem, identifying and obtaining necessary information has real meaning.

Estimation is almost entirely a mental activity. A 'point of reference' is useful as a visual or mental comparison to help in making estimates. For example, knowing and recognising your own height or the area of a football pitch can help in estimating other heights or areas. Pupils should be given opportunities to develop their own collection of 'points of reference'. With practice and experience of estimating they will refine and extend their references and improve their skills.

It is helpful to engage pupils in discussion about the size of the interval within which they are able to place an estimate and whether they have a useful point of reference. It is also helpful for pupils to consider their degree of confidence in their estimates, which will be affected by both of these factors. For example, a pupil may be able to say:

- I am 100 per cent confident that your height is between 1 m and 2 m because you are not unusually tall or small.
- I am 80 per cent confident that your height is between 1.6 m and 1.8 m because I am 1.55 m tall and you are a bit taller than I am.

In interpreting the suggested progression described below it should be recognised that estimation becomes more challenging if it is set within a context or uses units that are unfamiliar.

The *Framework for teaching mathematics: Years 7, 8 and 9*, supplement of examples, pages 230 to 231, provides further examples.

Make sensible estimates of a range of measures in relation to everyday situations	length, time
	mass
	area
	capacity, volume
	rates, e.g. *speed*

Rounding and continuity

Pupils should be helped to use their experience of the continuous number line to reinforce the continuous nature of many measures (including some that are often treated as discrete, such as age). A consequence is that measurements given to the nearest whole unit, for example, may be inaccurate by up to one half of the unit in either direction. It is important to take this inaccuracy into account when commenting constructively on the results obtained from calculations using these values.

The *Framework for teaching mathematics: Years 7, 8 and 9*, supplement of examples, page 231, provides further examples.

Appreciate the continuous nature of scales	
Appreciate imprecision of measurement	e.g. *How accurate is the area of a rectangle calculated to be 7.8 m² from lengths of 6.5 m and 1.2 m?*

Conversion

As pupils develop their skills beyond level 5 they should learn to use units and the symbols for units of measure systematically. It is important to consider the units that are most suitable to the context of a problem. It is also helpful to recognise that the conventions of unit notation can provide insights into the nature of the problem. For example:

- If pupils understand indices and reciprocals and can interpret fractions as division then they can see the structure of compound measures. For example, by interpreting km h^{-1} as km × 1/h and km ÷ h, they can connect the unit notation with the idea that speed is distance divided by time. This kind of connection can help with transformation of formulae.
- Conversion between different systems of measurement is usually undertaken by using approximate equivalence. For example, in converting from miles to metres one might take 1 mile to be approximately 1500 metres. By extension, to express a speed of 20 miles/hour (mph) in metres/hour (m/h) one might mentally calculate that at a speed of 20 miles/hour roughly 30 000 m would be travelled each hour.

Many conversion calculations involve making and using approximations. The same considerations apply to accuracy and rounding, as discussed above.

Know rough metric equivalents of imperial measures in daily use	8 km is approximately 5 miles
Convert one unit to another within a metric system	Metric length, mass and capacity e.g. *convert 750 g to kg* Metric area e.g. *convert 62 500 m² to hectares* Metric volume e.g. *convert 5.5 cm³ to mm³*
within a non-metric system	Time e.g. *approximately how many more minutes there are in January than February*
Between two systems	Currency e.g. *convert £12.67 to euros*
Understand and use compound measures	Convert one rate to another e.g. *convert 30 lb per square foot to kg/m²*

The *Framework for teaching mathematics: Years 7, 8 and 9*, supplement of examples, pages 228 to 229, provides further examples.

Mensuration

It is worth considering carefully how to deal with the formulae that are used in mensuration. Pupils are more likely to use and apply a formula with confidence if they can see how it links with the properties of the shape. This understanding may be triggered as pupils work together to deduce formulae and explain their features. In many cases the deduction will develop from knowledge of a more familiar shape. For example, pupils can extend knowledge about rectangles to derive a formula for the area of a parallelogram.

When a particular collection of formulae is established then pupils need practice in using the formulae to solve problems. There are several stages to this problem-solving process and the early and later stages are often given less attention than the central calculation.

- The early stages involve making decisions about which properties are significant and which formulae are appropriate. This is a mental stage and pupils benefit from being able to discuss the options. This can be supported by, for example, sorting and classifying questions into types.
- The later stages also involve decision-making, this time about checking the units and appropriate degrees of accuracy. Pupils should consider questions such as: 'How will the accuracy of input values affect the resulting solutions?'
- Pupils should also be encouraged to consider the links between a formula and the sort of units they might expect for the answer, and to check that they are consistent. For example, the formula $A = 2\pi r$ cannot be correct because A (area) requires square units and $2\pi r$ contains only one linear unit.

Perimeter

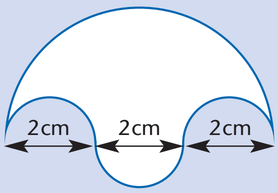
rectangle,

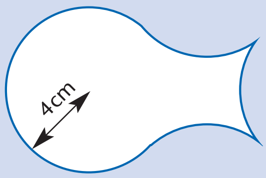
triangle,

compound shapes,

circle,

circular arcs

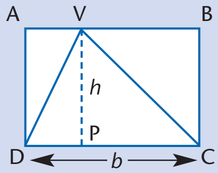


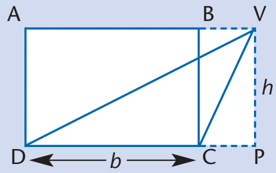


The *Framework for teaching mathematics: Years 7, 8 and 9*, supplement of examples, pages 234 to 236, provides further examples.

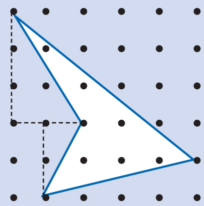
Area

Rectangle:
compound rectangles (bits in and bits out);
'adapted' rectangle (parallelogram, trapezium, right-angled triangle)

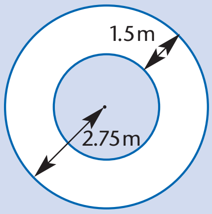


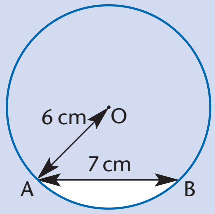


Triangle:
compound (bits in and bits out);
'adapted' triangle (rhombus, kite, trapezium, parallelogram, regular hexagon)



Circle:
compound (bits in and bits out);
'adapted' circle (semicircle, quadrant, sector, segment)



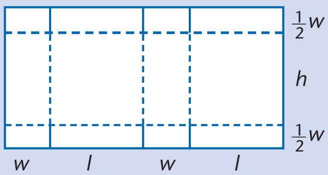


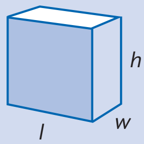
The *Framework for teaching mathematics: Years 7, 8 and 9*, supplement of examples, pages 234 to 237 provides further examples.

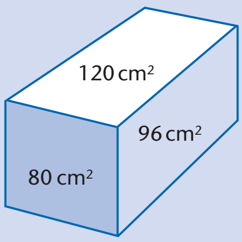
Volume and surface area

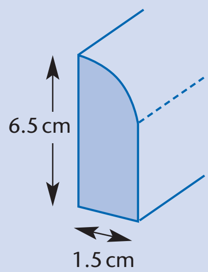
Identify and use the measurements necessary to calculate the volume and surface area of:

cube
cuboid
right prisms including cylinder
cone and sphere
compound 3-D shapes









The *Framework for teaching mathematics: Years 7, 8 and 9*, supplement of examples, pages 238 to 241, provides further examples.

Compound measure

The calculation stages of a problem involving compound measures can cause difficulties if pupils do not give some time to the mental stages at the start. At the outset it is important to establish a clear rationale for the units in the question and how these link together. This clarity can help to inform the various stages of any calculation and the final form of the solution. To develop this habit of mind, it can be worthwhile concentrating only on this 'entry stage' for a collection of problems. The solution may not be completed but pupils should verbalise the units and identify the link between these units and the nature of the measures in the calculation and solution. For example, they might identify the stages and associated units in working out the cost of petrol for a journey of 140 miles if the minibus averages 11 miles to the litre and petrol costs 90 pence per litre.

It is through a more thorough understanding of the connections between all the units in a mensuration problem that pupils will be better able to determine the dimension of formulae from inspecting the nature of the variables involved. This understanding will also reinforce accuracy when pupils are required to change the subject of a mensuration formula, for example, amount of petrol = rate of consumption × distance travelled.

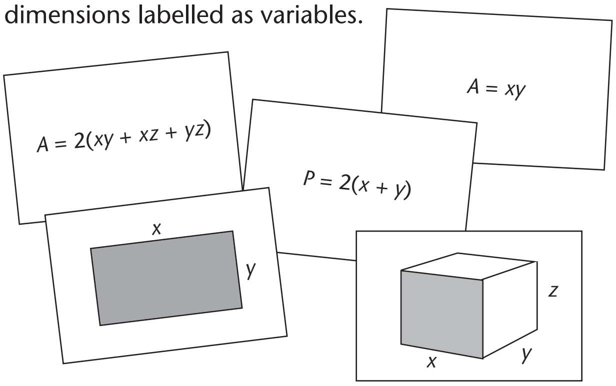
Understand and use compound measure, for example, speed, density or pressure to solve problems
Petrol consumption 60 litres per km
Speed of Pluto's orbit 1.06×10^4 mph
Density of gold 19.3 g/cm^3
Atmospheric pressure 1.013×10^5 N/m^2

The *Framework for teaching mathematics: Years 7, 8 and 9*, supplement of examples, page 233 and page 21, provides further examples.

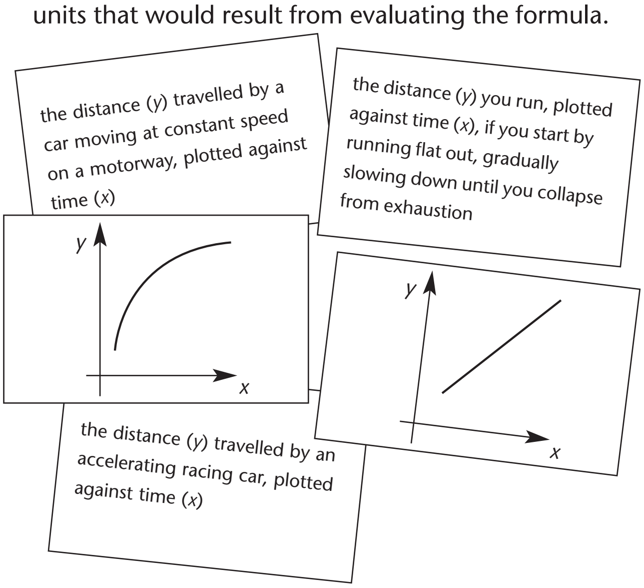
Classifying cards in which, for example, pupils sort cards according to the dimensions of expressions or formulae, will help build their understanding of the connections within formulae of the same dimension and between formulae of different dimensions. A set of cards could include formulae for perimeters or other lengths, areas and surface areas, volumes and compound measures such as rates. Look for examples from other areas of the curriculum that may be relevant to the pupils' year group. It is possible that pupils will be familiar with using such formulae but may not have considered their mathematical structure.

For an illustration of the types of formulae see the *Framework for teaching mathematics: Year 7, 8 and 9,* supplement of examples, pages 138 to 141.

Matching different forms of representation is also helpful. For example, pupils could be asked to match a set of cards showing mensuration formulae for perimeter, area, surface area and volume to another set showing diagrams of 2-D and 3-D shapes with dimensions labelled as variables.



The task could be extended by asking pupils to suggest units for the dimensions and thus agree the units that would result from evaluating the formula.

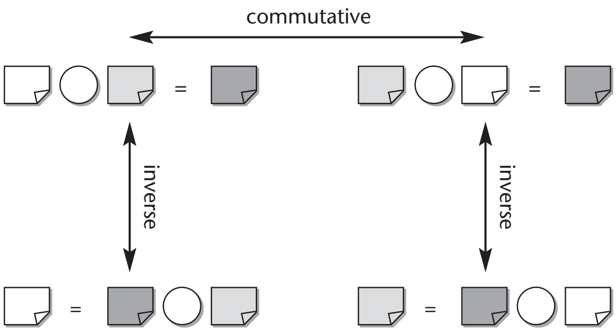


Alternatively, pupils could be asked to match some cards showing graphs to others giving descriptions of the graphs.

The graphs can also be used to develop understanding of compound measures (rates); pupils should be encouraged to attach units to the variables and to discuss the rates of change shown in each graph.

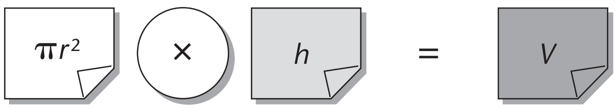
For an illustration of graphs and descriptions see the *Framework for teaching mathematics: Years 7, 8 and 9,* supplement of examples, page 175.

Family of formulae is an activity using a template for the elements and operations in a formula as shown below.



Use colour-coded sticky notelets to hide the three 'objects' and the operations (use circles) needed to make a family of formulae. Show one of the formulae and use pupils' knowledge of arithmetical relationships (families of facts) to generate the three related formulae.

It is important to consider more complex items as objects in this structure so, for example, an object might be πr^2.



It is important that pupils remember the structure of these relationships and can recall it as a mental image.

Pupils' understanding is strengthened by the regular layout, use of colour, 'hide and reveal' tasks, using their own cards (moving and annotating) and as much talk and shared explanation as possible. This can help pupils to realise how simple it can be to change the subject of a formula if the right term (which might be a collection of variables) is considered as a single element or object.

Annotate written solutions requires pupils to compare ways of writing solutions. Provide a selection of very detailed written solutions to the same mensuration problem and ask pupils to highlight those stages they would confidently omit when writing out the solution. Ask pairs of pupils to compare the mental stages and to explain how they mentally process the steps. Give each pair another mensuration problem to solve and ask them to give each other a commentary on what they write and what they think, with particular emphasis on the units associated with the figures that they write or say.

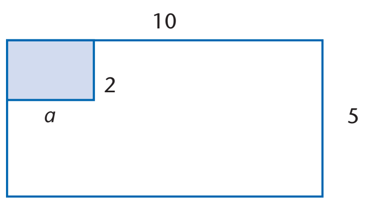
Discussing strategies in this way stimulates pupils to think about their thinking (metacognition). Following collaborative work of this sort, many pupils will refine their own processes. Forcing the discussion to focus on the relationship between the units in the problem can help pupils to develop clearer methods and avoid common errors.

Say what you see, an activity in deducing formulae for perimeters, areas and volumes of complex shapes, can require pupils to see how a shape can be 'cut up into' or 'built up from' various parts. For example, using methods such as a 'dissection' or 'boxing' can generate algebraic expressions to represent an area. Working with shapes in this way can help pupils to appreciate the equivalence of expressions representing the same area and improve the range of strategies they employ when visualising calculations. Ask pupils to express the area 'in as many (essentially different) ways as you can'. For example:

$50 - 2a$

$2(10 - a) + 30$

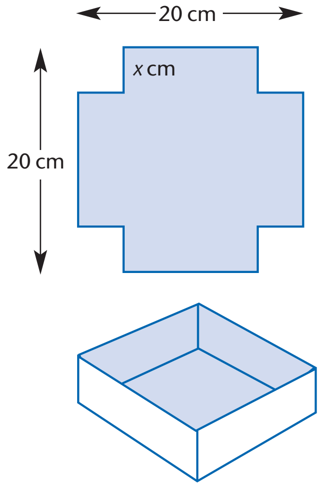
$3a + 5(10 - a)$



For further examples see the *Framework for teaching mathematics: Years 7, 8 and 9*, supplement of examples, pages 119 and 121.

Living graphs is an activity in which prepared images prompt pupils' thinking around a real-life context. It leads to pupils composing their own interpretations and ultimately constructing a graph for themselves. An example, *The speed of a racing car*, is given on page 32 of *Teaching mental mathematics from level 5: algebra* (DfES 1287-2005).

This idea can be used to develop activities linking properties of shapes to formulae such as those for volume and surface area, and to provide a context in which pupils can connect formulae to words, pictures, tables and graphs. For example, a large graph could be prepared, showing volume as a function of x for an open box with height x (see diagram below).



'Statement cards' could show nets with the size of the cut-out shown, values of x (for example, $x = 3$) and small extracts from a spreadsheet showing length, width and height of the open box.

Pupils have to decide where the cards would best be positioned on the graph from the information they have been given and justify their reasoning to the group.

In this way pupils build a mental picture of the links between a table or function and the graph, and between the real-life context and sections of the graph. Pupils may go on to create additional cards to describe:

- ☐ other sections or points on the graph;
- ☐ rates of change on the graph.

Graphing the changes that occur when 2-D and 3-D shapes are transformed in some way gives pupils a real context in which to discuss how one variable relates to another. The discussion does not involve calculation or plotting but focuses on the relationship between the variables as the shape changes.

For a 3-D example, ask pupils to imagine a **cuboid** lying with its length flat on the desk, then to imagine it being stretched along one axis parallel to the desk-top. Ask:

- ☐ What happens to the height?
- ☐ What happens to the length?

Check that pupils understand that stretching is a mathematical term that, in this case, involves changes in one dimension only. A common misconception is to think of a 'stretchy' shape as one in which the height reduces as the length increases.

Continue with the investigation, asking pupils to sketch the graph of:

- ☐ the height against the length;
- ☐ the volume against the height;
- ☐ the volume against the length.

For a 2-D example, ask pupils to imagine a **large circle** with a fixed radius, then to imagine a smaller circle with the same centre and variable radius. Ask:

- ☐ What happens to the area between the circles as the smaller circle gets bigger?
- ☐ How does the area of the large circle compare to the area of the small circle at the point when the smaller circle has a radius that is half that of the bigger radius?
- ☐ Sketch the radius of the small circle against the area between the circles…

Thinking about variables in this way is a useful precursor to some of the generalisations pupils need to understand in order to carry out investigations in shape, space and measures.

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