WG

A WHITE GROUP MATHEMATICS PUBLICATION

(www.whitegroupmaths.com)

SIXTH TERM EXAMINATION PAPERS WORKED PROBLEMS

VOLUME 1

A FEW WORDS

The amount of Sixth Term Examination Papers (STEP) resources on the World Wide Web are far and few, hence my motivation to produce this volume to prepare the student more adequately for this exceptionally demanding examination standard. All solutions were personally (painstakingly if I may add) drafted by myself- a process which took me several months. Not to mention numerous hours spent on the careful selection of questions. Kindly note that the worked problems presented here belong to difficulty level III; the actual question texts are not reproduced in consideration of copyright matters. It is my sincere hope that you will find this compilation a worthy study-aid. Good luck and god bless.

Humbly,

Frederick Koh

BEng (Hons)

BSc (Hons)

Note: This is a short version of the preview; for a longer version head over to

http://www.facebook.com/whitegroupmathematics (Click on "FOR FANS ONLY" tab).

The full version of this volume will be out later in 2013.

STEP III, 2000 Question 2

SOLUTIONS :

For the substitution $x = 2 - \cos\theta$, $\dfrac{dx}{d\theta} = \sin\theta \Rightarrow dx = \sin\theta \; d\theta$

When $x = 2$, $\theta = \dfrac{\pi}{2}$; $x = \dfrac{3}{2}$, $\theta = \dfrac{\pi}{3}$

$$\int_{\frac{3}{2}}^{2} \left(\frac{x-1}{3-x}\right)^{\frac{1}{2}} dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} (\sin\theta) d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \sqrt{\frac{1-\cos\theta}{1-\cos\theta}} (\sin\theta) d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{(1-\cos\theta)}{\sqrt{1-\cos^2\theta}} (\sin\theta) d\theta = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{(1-\cos\theta)}{\sin\theta} (\sin\theta) d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 1 - \cos\theta \; d\theta \; = [\theta - \sin\theta]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \left(\frac{\pi}{2} - 1\right) - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1 \quad \text{(shown)}$$

The appropriate substitution would be $x = \left(\dfrac{a+b}{2}\right) - \left(\dfrac{b-a}{2}\right)\cos\theta$,

where $\dfrac{dx}{d\theta} = \left(\dfrac{b-a}{2}\right)\sin\theta \Rightarrow dx = \left(\dfrac{b-a}{2}\right)\sin\theta \; d\theta$

When $x = q = \dfrac{a+b}{2}$, $\theta = \dfrac{\pi}{2}$; $x = p = \dfrac{3a+b}{4}$, $\theta = \dfrac{\pi}{3}$

Then $$\int_p^q \left(\frac{x-a}{b-x}\right)^{\frac{1}{2}} dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{\frac{\left(\frac{a+b}{2}\right) - \left(\frac{b-a}{2}\right)\cos\theta - a}{b - \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)\cos\theta}} \left(\left(\frac{b-a}{2}\right)\sin\theta\right) d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{\frac{\left(\frac{b-a}{2}\right) - \left(\frac{b-a}{2}\right)\cos\theta}{\left(\frac{b-a}{2}\right) + \left(\frac{b-a}{2}\right)\cos\theta}} \left(\left(\frac{b-a}{2}\right)\sin\theta\right) d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{\frac{\left(\frac{b-a}{2}\right) - \left(\frac{b-a}{2}\right)\cos\theta}{\left(\frac{b-a}{2}\right) + \left(\frac{b-a}{2}\right)\cos\theta}} \sqrt{\frac{\left(\frac{b-a}{2}\right) - \left(\frac{b-a}{2}\right)\cos\theta}{\left(\frac{b-a}{2}\right) - \left(\frac{b-a}{2}\right)\cos\theta}} \left(\left(\frac{b-a}{2}\right)\sin\theta\right) d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\left(\frac{b-a}{2}\right) - \left(\frac{b-a}{2}\right)\cos\theta}{\sqrt{\left(\frac{b-a}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2 \cos^2\theta}} \left(\left(\frac{b-a}{2}\right)\sin\theta\right) d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\left(\frac{b-a}{2}\right) - \left(\frac{b-a}{2}\right)\cos\theta}{\left(\left(\frac{b-a}{2}\right)\sin\theta\right)} \left(\left(\frac{b-a}{2}\right)\sin\theta\right) d\theta$$

$$= \left(\frac{b-a}{2}\right) \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 1 - \cos\theta \; d\theta \;= \left(\frac{b-a}{2}\right)\left[\theta - \sin\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \left(\frac{b-a}{2}\right)\left[\left(\frac{\pi}{2} - 1\right) - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right]$$

$$= \left(\frac{b-a}{2}\right)\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1\right) \;= \left(\frac{b-a}{2}\right)\left(\frac{\pi + 3\sqrt{3} - 6}{6}\right)$$

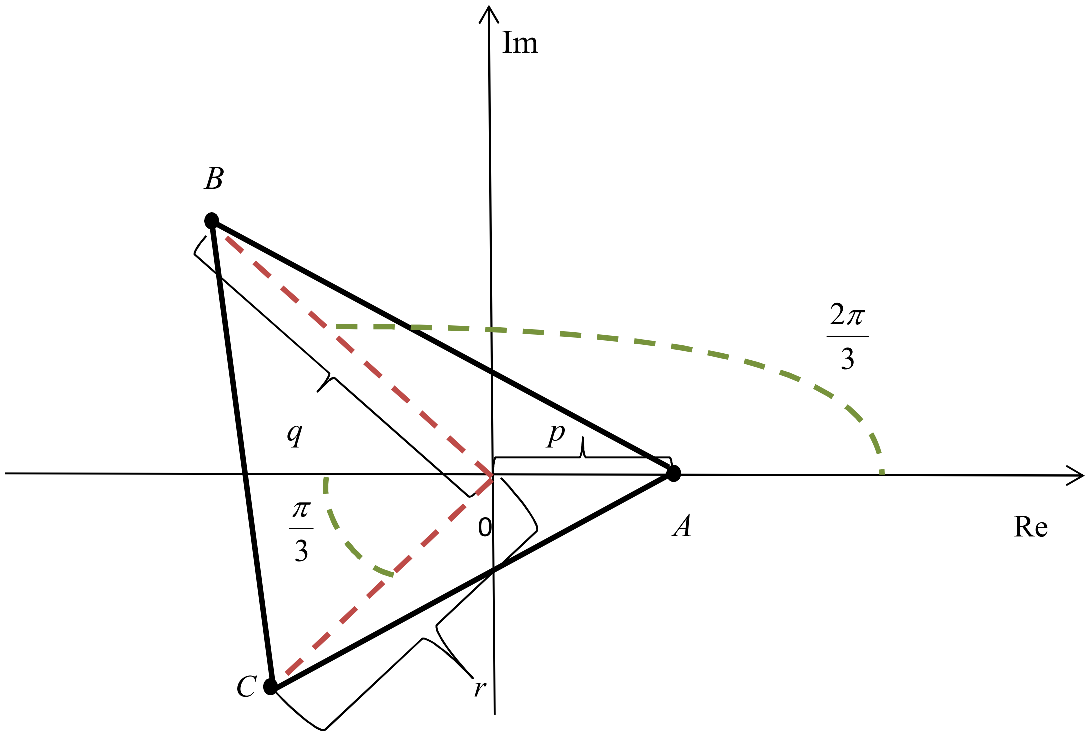
$$= \frac{(b-a)\left(\pi + 3\sqrt{3} - 6\right)}{12} \quad \text{(shown)}$$

STEP III, 2000 Question 3

SOLUTIONS :

$$1 + \alpha^2 = 1 + e^{\frac{2\pi}{3}i} = 1 + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \left(1 - \frac{1}{2}\right) + \frac{\sqrt{3}}{2}i$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{\frac{\pi}{3}i} = \alpha \quad \text{(shown)}$$



$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=-r\alpha-p$

$\overrightarrow{AN}=(-r\alpha-p)e^{i\frac{\pi}{3}}=(-r\alpha-p)\alpha=-r\alpha^2-p\alpha$

$\therefore\ \overrightarrow{ON}=\overrightarrow{OA}+\overrightarrow{AN}=p-r\alpha^2-p\alpha=(1-\alpha)p-r\alpha^2$ (shown)

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=q\alpha^2-p$

$\overrightarrow{LB}=(q\alpha^2-p)e^{i\frac{\pi}{3}}=(q\alpha^2-p)\alpha=q\alpha^3-p\alpha=-q-p\alpha$ $[\because\ \alpha^3=e^{i\pi}=-1]$

$\therefore\ \overrightarrow{OL}=\overrightarrow{OB}-\overrightarrow{LB}=q\alpha^2+q+p\alpha=q(\alpha^2+1)+p\alpha=q\alpha+p\alpha$ (shown)

$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=-r\alpha-q\alpha^2$

$\overrightarrow{BM}=(-r\alpha-q\alpha^2)e^{-i\frac{\pi}{3}}=(-r\alpha-q\alpha^2)\left(\frac{1}{\alpha}\right)=-r-q\alpha$

$\therefore\ \overrightarrow{OM}=\overrightarrow{OB}+\overrightarrow{BM}=q\alpha^2-r-q\alpha=-r-q(\alpha-\alpha^2)=-r-q$ (shown)

Argument of complex number $N=\arg\left[(1-\alpha)p-r\alpha^2\right]=\arg\left[(1-1-\alpha^2)p-r\alpha^2\right]$

$=\arg\left(-\alpha^2p-r\alpha^2\right)=\arg\left[-(p+r)\alpha^2\right]$

$$= \arg\left(-\alpha^2\right) = \arg(-1) + \arg(\alpha^2) = \pi + \frac{2\pi}{3} = \frac{5\pi}{3} = -\frac{\pi}{3}$$

Hence, NB is a continuous straight line passing through the origin.

(By understanding the above functions on the fact that the chords ON and OB are inclined at the same angle to the horizontal as well as the vertical, though proving the horizontal part indirectly also implies the vertical part is true)

Argument of complex number $L = \arg(q\alpha + p\alpha) = \arg(\alpha) = \frac{\pi}{3}$

Hence, LC is a continuous straight line passing through the origin.

Argument of complex number $M = \arg(-r - q) = \pi$

Hence, MA is a continuous flat horizontal line(lying on the real axis) passing through the origin.

Reconciling all 3 observations gives the fact that lines LC, MA and NB meet at the origin. (shown)

$|\overrightarrow{LC}| = |-r\alpha - q\alpha - p\alpha| = |\alpha(p+q+r)| = p+q+r \quad [\because\ |\alpha| = 1]$ (shown)

$|\overrightarrow{MA}| = |p - (-r - q)| = |p+q+r| = p+q+r$ (shown)

$|\overrightarrow{NB}| = |q\alpha^2 - \left(-\alpha^2 p - r\alpha^2\right)| = |\alpha^2(p+q+r)| = p+q+r \quad [\because\ |\alpha^2| = 1]$ (shown)

STEP III, 2000 Question 8

<u>SOLUTIONS :</u>

Let P_n be the proposition that $a_n = 3a_{n-1} - a_{n-2}$, where $a_n = \frac{1+a_{n-1}^2}{a_{n-2}}$, $a_0 = a_1 = 1$ and $n \geq 2$

For P_2, $\quad a_2 = 3a_1 - a_0 = 3 - 1 = 2 \quad$ and $\quad a_2 = \frac{1+a_1^2}{a_0} = \frac{1+1^2}{1} = 2$

Since LHS=RHS, P_2 is true.

Assume P_{k-1}, P_k are both true for some positive integer value of k where $k \geq 2$,

ie $\quad a_{k-1} = 3a_{k-2} - a_{k-3}$ and $a_k = 3a_{k-1} - a_{k-2}$

Considering $P_{k+1}: a_{k+1} = \dfrac{1+a_k^2}{a_{k-1}} = \dfrac{1+\left(3a_{k-1}-a_{k-2}\right)^2}{a_{k-1}} = \dfrac{1+9a_{k-1}^2-6a_{k-1}a_{k-2}+a_{k-2}^2}{a_{k-1}}$

$$= \frac{1+a_{k-2}^2}{a_{k-1}} + 9a_{k-1} - 6a_{k-2}$$

$$= a_{k-3} + 9a_{k-1} - 6a_{k-2}$$

$$= a_{k-3} + 3(a_k + a_{k-2}) - 6a_{k-2}$$

$$= a_{k-3} + 3a_k - 3a_{k-2}$$

$$= 3a_k - \left(3a_{k-2} - a_{k-3}\right)$$

$$= 3a_k - a_{k-1}$$

P_{k-1}, P_k are true $\Rightarrow P_{k+1}$ is true.

Since P_2 is true, by mathematical induction, P_n is true for all $n \ge 2$. (shown)

The **difference equation** is given by $a_n - 3a_{n-1} + a_{n-2} = 0,\ \ n \ge 2$

Auxiliary equation is $m^2 - 3m + 1 = 0 \Rightarrow m = \dfrac{3 \pm \sqrt{5}}{2}$

Hence, general solution is given by $a_n = A\left(\dfrac{3+\sqrt{5}}{2}\right)^n + B\left(\dfrac{3-\sqrt{5}}{2}\right)^n,\ n \ge 1$

Recognising that $\dfrac{3+\sqrt{5}}{2} = 1 + \dfrac{1+\sqrt{5}}{2} = 1 + \alpha$,

and $\dfrac{3-\sqrt{5}}{2} = 1 - \dfrac{(\sqrt{5}-1)}{2} = 1 - \dfrac{2}{1+\sqrt{5}} = 1 - \dfrac{1}{\frac{1+\sqrt{5}}{2}} = 1 - \dfrac{1}{\alpha}$, where $\alpha = \dfrac{1+\sqrt{5}}{2}$,

The general solution can therefore be rewritten as $a_n = A(1+\alpha)^n + B\left(1-\dfrac{1}{\alpha}\right)^n$

When as $n = 1$, $a_1 = 1 = A(1+\alpha) + B\left(1-\dfrac{1}{\alpha}\right) = A(1+\alpha) + B\left(\dfrac{\alpha-1}{\alpha}\right)$ -------- (1)

When as $n=2,\ a_2=2=A(1+\alpha)^2+B\left(1-\frac{1}{\alpha}\right)^2=A(1+\alpha)^2+B\left(\frac{\alpha-1}{\alpha}\right)^2$ -------- (2)

$$(1)\times-\left(\frac{\alpha-1}{\alpha}\right)+(2):\ A\left[\left(\frac{1-\alpha^2}{\alpha}\right)+(1+\alpha)^2\right]=-\left(\frac{\alpha-1}{\alpha}\right)+2$$

$$A\left[\frac{1}{\alpha}-\alpha+1+2\alpha+\alpha^2\right]=\left(\frac{1-\alpha}{\alpha}\right)+2$$

$$A\left[\frac{1}{\alpha}+1+\alpha+\alpha^2\right]=\frac{1-\alpha+2\alpha}{\alpha}$$

$$A\left[\frac{1}{\alpha}(1+\alpha^2)+1+\alpha^2\right]=\frac{1+\alpha}{\alpha}$$

$$A\left[\left(\frac{1}{\alpha}+1\right)(1+\alpha^2)\right]=\frac{1+\alpha}{\alpha}$$

$$A\left[\left(\frac{1+\alpha}{\alpha}\right)(1+\alpha^2)\right]=\frac{1+\alpha}{\alpha}$$

$$\therefore\ A=\frac{1}{1+\alpha^2}$$

$$(1)\times-(1+\alpha)+(2):\ B\left[-\left(\frac{\alpha^2-1}{\alpha}\right)+\left(\frac{\alpha-1}{\alpha}\right)^2\right]=-(1+\alpha)+2$$

$$B\left[\left(\frac{1-\alpha^2}{\alpha}\right)+\left(\frac{\alpha^2-2\alpha+1}{\alpha^2}\right)\right]=1-\alpha$$

$$B\left[\frac{1}{\alpha}-\alpha+1-\frac{2}{\alpha}+\frac{1}{\alpha^2}\right]=1-\alpha$$

$$B\left[-\alpha+1-\frac{1}{\alpha}+\frac{1}{\alpha^2}\right]=1-\alpha$$

$$B\left[(-\alpha+1)+\frac{1}{\alpha^2}(-\alpha+1)\right]=1-\alpha$$

$$B\left[1+\frac{1}{\alpha^2}\right]=1$$

$$B\left[\frac{\alpha^2+1}{\alpha^2}\right]=1$$

$$\therefore\ B=\frac{\alpha^2}{1+\alpha^2}$$

Substituting these into the general solution gives

$$a_n=\left(\frac{1}{1+\alpha^2}\right)(1+\alpha)^n+\left(\frac{\alpha^2}{1+\alpha^2}\right)\left(\frac{\alpha-1}{\alpha}\right)^n$$

$$\frac{1}{1+\alpha^2}\left[(1+\alpha)^n+\alpha^2\left(\frac{\alpha-1}{\alpha}\right)^n\right]\text{---------(3)}$$

Before proceeding, certain results must be made available to simplify (3):

$$\alpha^2=\left(\frac{1+\sqrt{5}}{2}\right)^2=\frac{6+2\sqrt{5}}{4}=\frac{3+\sqrt{5}}{2}=1+\alpha$$

$$\frac{\alpha-1}{\alpha}=\frac{(\alpha-1)(\alpha+1)}{\alpha(\alpha+1)}=\frac{\alpha^2-1}{\alpha(\alpha+1)}=\frac{\alpha}{\alpha(\alpha^2)}=\alpha^{-2}$$

$$\frac{1}{1+\alpha^2}=\frac{1}{1+\left(\frac{3+\sqrt{5}}{2}\right)}=\frac{2}{5+\sqrt{5}}=\frac{1}{\sqrt{5}}\left(\frac{2}{1+\sqrt{5}}\right)=\frac{1}{\sqrt{5}\alpha}$$

Hence, (3) becomes

$$a_n=\frac{1}{\sqrt{5}\alpha}\left[(\alpha^2)^n+\alpha^2(\alpha^{-2})^n\right]$$

$$=\frac{1}{\sqrt{5}\alpha}\left[(\alpha^2)^n+\alpha^2(\alpha^{-2})^n\right]$$

$$=\frac{1}{\sqrt{5}\alpha}\left[\alpha^{2n}+\alpha^{2-2n}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\alpha^{2n-1}+\alpha^{1-2n}\right]$$

$$=\frac{\alpha^{2n-1}+\alpha^{-(2n-1)}}{\sqrt{5}}\quad\text{(shown)}$$

STEP III, 1998 Question 8

<u>SOLUTIONS :</u>

(i) When the sphere intersects the line, $(b+\lambda m)\bullet(b+\lambda m)=a^2$

$$b\bullet b+2\lambda(b\bullet m)+\lambda^2(m\bullet m)=a^2$$

$$\lambda^2(m\bullet m)+2\lambda(b\bullet m)+(b\bullet b-a^2)=0$$

Where we have a quadratic equation in λ.

Hence, if the two distinct points of intersection occur, then

$$(2b\bullet m)^2-4(m\bullet m)(b\bullet b-a^2)>0$$

$$4(b\bullet m)^2-4(m\bullet m)(b\bullet b-a^2)>0$$

$$(b\bullet m)^2-(m\bullet m)(b\bullet b-a^2)>0$$

Since m is a unit vector, then $m\bullet m=|m|^2=1$ and the inequality is further reduced to

$$(b\bullet m)^2-(b\bullet b-a^2)>0$$

$$(b\bullet m)^2-b\bullet b+a^2>0$$

$\therefore\quad a^2>b\bullet b-(b\bullet m)^2$ (shown)

When only one point of intersection occurs, then $(2b\bullet m)^2-4(m\bullet m)(b\bullet b-a^2)=0$

and similar further simplification would give rise to $a^2=b\bullet b-(b\bullet m)^2$ (shown)

For this situation, $\lambda=\dfrac{-2b\bullet m}{2m\bullet m}=-b\bullet m$

and $p=b+(-b\bullet m)m$ (substituting the above expression for λ into the line $r=b+\lambda m$ gives point P)

$$p=b-(b\bullet m)m$$

$$p\bullet m=[b-(b\bullet m)m]\bullet m$$

$$p\bullet m=[b-(b\bullet m)m]\bullet m$$

$p\bullet m=b\bullet m-(b\bullet m)(m\bullet m)=b\bullet m-b\bullet m=0$ (shown)

(ii) If the circle is tangential to the plane, then only one point of intersection exists. If we let this point have position vector q, we have $q - d = an$.

Equation of plane is $r \bullet n = l$; since q clearly lies on the plane,

$$q \bullet n = l$$

Substituting the above equation into this gives $(an + d) \bullet n = l$

$$a(n \bullet n) + d \bullet n = l$$

$$a + d \bullet n = l$$

$$a = l \text{ (shown)}$$

(Note: $d \bullet n$ =0 because the normal of the tangential plane would be perpendicular to the radius vector through the centre of the second circle)

(iii) Equation of second sphere is given by $(r - d) \bullet (r - d) = a^2$

When the two spheres intersect, $(r - d) \bullet (r - d) = r \bullet r$

$$r \bullet r - 2r \bullet d + d \bullet d = r \bullet r$$

$$r \bullet d = \frac{1}{2}(d \bullet d)$$

Let a possible point of intersection(out of the infinite number of points) have position vector = c

In order for the circles to intersect in a perpendicular configuration,

$$c \bullet (c - d) = 0 ------(3) \quad \text{and}$$

$$c \bullet d = \frac{1}{2}(d \bullet d) ------(4),$$

From (3), we have $c \bullet c = c \bullet d$; substituting this into (4) gives $c \bullet c = \frac{1}{2}(d \bullet d)$

Since this point of intersection also lies on the first circle, $c \bullet c = a^2$

and therefore $\frac{1}{2}(d \bullet d) = a^2 \Rightarrow d \bullet d = 2a^2$ (shown)

STEP III, 1998 Question 5

SOLUTIONS :

$$M^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -I \text{ (shown)}$$

$$\exp(\theta M) = \sum_{r=0}^{\infty} \frac{1}{r!}(\theta M)^r = I + \theta M + \frac{1}{2!}\theta^2 M^2 + \frac{1}{3!}\theta^3 M^3 + \frac{1}{4!}\theta^4 M^4 + \dots\dots\dots\dots$$

$$= I + \theta M - \frac{1}{2!}\theta^2 I - \frac{1}{3!}\theta^3 M + \frac{1}{4!}\theta^4 I + \dots\dots\dots$$

$$= I\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\dots\dots\right) + M\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\dots\dots\right)$$

$= I\cos\theta + M\sin\theta$ (replacing both series by their Maclaurin's equivalents)

$$= \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix} + \begin{pmatrix} 0 & -\sin\theta \\ \sin\theta & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ (shown)}$$

$\exp(\theta M)$ represents a transformation matrix which serves to **rotate** any given set of coordinates(in the $x - y$ plane) by an angle of θ about the origin in the **clockwise** direction. (shown)

$$N^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{(null matrix)}$$

$$\Rightarrow N^3 = N^4 = N^5 = \dots\dots\dots = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\exp(sN) = \sum_{r=0}^{\infty} \frac{1}{r!}(sN)^r = I + sN = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

$\exp(sN)$ represents a transformation matrix which serves to **shear** any given set of coordinates parallel to the x axis such that the original x coordinate x_1 is adjusted to $x_1 + sy_1$ while leaving the y coordinate y_1 unchanged. (shown)

When $\exp(sN)\exp(\theta M) = \exp(\theta M)\exp(sN)$,

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta + s\sin\theta & s\cos\theta - \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & s\cos\theta - \sin\theta \\ \sin\theta & s\sin\theta + \cos\theta \end{pmatrix}$$

By observation, $\theta = n\pi$, where $n \in \mathrm{Z}$ (shown)

The sequence of transformations (rotation and shearing) is **commutative** only IF the angle of rotation θ is an integer multiple of π. (shown)

By considering integration by parts and chain recurrence, show that

$$I(a,b) = \int_0^1 t^a (1-t)^b \, dt = \frac{a!b!}{(a+b+1)!} \quad \text{for } a \geq 0, \; b \geq 0$$

(Note: I have modified this question substantially from its original version.)

SOLUTION :

$$I(a,b) = \int_0^1 t^a (1-t)^b \, dt = \left[\frac{t^{a+1}}{(a+1)} (1-t)^b \right]_0^1 + \frac{b}{a+1} \int_0^1 t^{a+1} (1-t)^{b-1} \, dt$$

$$= \frac{b}{a+1} \int_0^1 t^{a+1} (1-t)^{b-1} \, dt = \frac{b}{a+1} I(a+1, b-1)$$

$$= \left(\frac{b}{a+1}\right)\left(\frac{b-1}{a+2}\right) I(a+2, b-2) = \left(\frac{b}{a+1}\right)\left(\frac{b-1}{a+2}\right)\left(\frac{b-2}{a+3}\right) I(a+3, b-3)$$

$$= \left(\frac{b}{a+1}\right)\left(\frac{b-1}{a+2}\right)\left(\frac{b-2}{a+3}\right) \cdots\cdots\cdots \left(\frac{1}{a+b}\right) . I(a+b, 0)$$

$$= \frac{a!}{a!}\left[\left(\frac{b}{a+1}\right)\left(\frac{b-1}{a+2}\right)\left(\frac{b-2}{a+3}\right) \cdots\cdots\cdots \left(\frac{1}{a+b}\right)\right] I(a+b, 0)$$

$$= \frac{a!b!}{(a+b)!} I(a+b, 0) = \frac{a!b!}{(a+b)!} \int_0^1 t^{a+b} \, dt$$

$$= \frac{a!b!}{(a+b)!(a+b+1)} = \frac{a!b!}{(a+b+1)!} \quad \text{(shown)}$$

STEP III, 1999 Question 1

SOLUTIONS :

(i) $\sum \alpha = p \Rightarrow ak^{-1} + a + ak = p$

$$a(k^{-1} + 1 + k) = p - - - - - - - (1)$$

$$\sum \alpha\beta = q \Rightarrow (ak^{-1})(a) + (a)(ak) + (ak)(ak^{-1}) = q$$

$$a^2k^{-1} + a^2k + a^2 = q$$

$$a^2(k^{-1} + 1 + k) = q - - - - - - - (2)$$

$$\sum \alpha\beta\gamma = r \Rightarrow (ak^{-1})(a)(ak) = r$$

$$a^3 = r - - - - - - - (3)$$

$(2) \div (1)$: $a = \dfrac{q}{p}$ (shown)

Substituting this into (3): $\left(\dfrac{q}{p}\right)^3 = r \Rightarrow q^3 = rp^3$

$\therefore \quad q^3 - rp^3 = 0$ (shown)

(ii) $\sum \alpha = p \Rightarrow \alpha + \beta + \gamma = p$

$$\alpha + \beta = p - \gamma - - - - - - - - - - - (1)$$

(where α, β and γ are generic roots of the equation)

$$\sum \alpha\beta = q \Rightarrow \alpha\beta + \beta\gamma + \alpha\gamma = q$$

$$\alpha\beta + \gamma(\alpha + \beta) = q - - - - - - - (2)$$

$$\sum \alpha\beta\gamma = r = \frac{q^3}{p^3} \Rightarrow \alpha\beta\gamma = \frac{q^3}{p^3}$$

$$\alpha\beta = \frac{q^3}{p^3\gamma} - - - - - - - - - - - - (3)$$

Substitute both (1) and (3) into (2):

$$\frac{q^3}{p^3\gamma}+\gamma(p-\gamma)=q$$

$$q^3+p^3\gamma^2(p-\gamma)=qp^3\gamma$$

$$q^3+p^4\gamma^2-p^3\gamma^3=qp^3\gamma$$

$$p^3\gamma^3-p^4\gamma^2+qp^3\gamma-q^3=0$$

Let $f(\gamma)=p^3\gamma^3-p^4\gamma^2+qp^3\gamma-q^3$

$$f\left(\frac{q}{p}\right)=p^3\left(\frac{q}{p}\right)^3-p^4\left(\frac{q}{p}\right)^2+qp^3\left(\frac{q}{p}\right)-q^3=q^3-p^2q^2+p^2q^2-q^3=0$$

Hence, $\gamma=\frac{q}{p}$ is a root of the equation $x^3-px^2+qx-r=0$ (shown)

Substituting $\gamma=\frac{q}{p}$ into (3) gives $\alpha\beta=\left(\frac{q^3}{p^3}\right)\left(\frac{p}{q}\right)=\frac{q^2}{p^2}$ (shown)

An obvious relationship arising is $\gamma^2=\alpha\beta$

where $\frac{\gamma}{\alpha}=\frac{\beta}{\gamma}=$ common ratio

Hence, the 3 roots are in geometric progression .(Note though that the sequencing of the roots to form a proper GP in my solving context is in the following order: $\alpha,\ \gamma,\ \beta$) (shown)

(iii) Let the 3 roots be $a-d,\ a,\ a+d$

Then $\sum\alpha=p\Rightarrow 3a=p$, ie $a=\frac{p}{3}$

$$\sum\alpha\beta=q\Rightarrow(a-d)(a)+(a)(a+d)+(a+d)(a-d)=q$$

$$a^2-ad+a^2+ad+a^2-d^2=q$$

$$3a^2-d^2=q$$

$$3\left(\frac{p^2}{9}\right)-d^2=q \qquad [\ \because a=\frac{p}{3}]$$

$$d^2=\frac{p^2}{3}-q$$

$$\sum \alpha\beta\gamma = r \Rightarrow (a-d)(a)(a+d) = r$$

$$a^3 - ad^2 = r$$

$$\left(\frac{p}{3}\right)^3 - \left(\frac{p}{3}\right)\left(\frac{p^2}{3} - q\right) = r$$

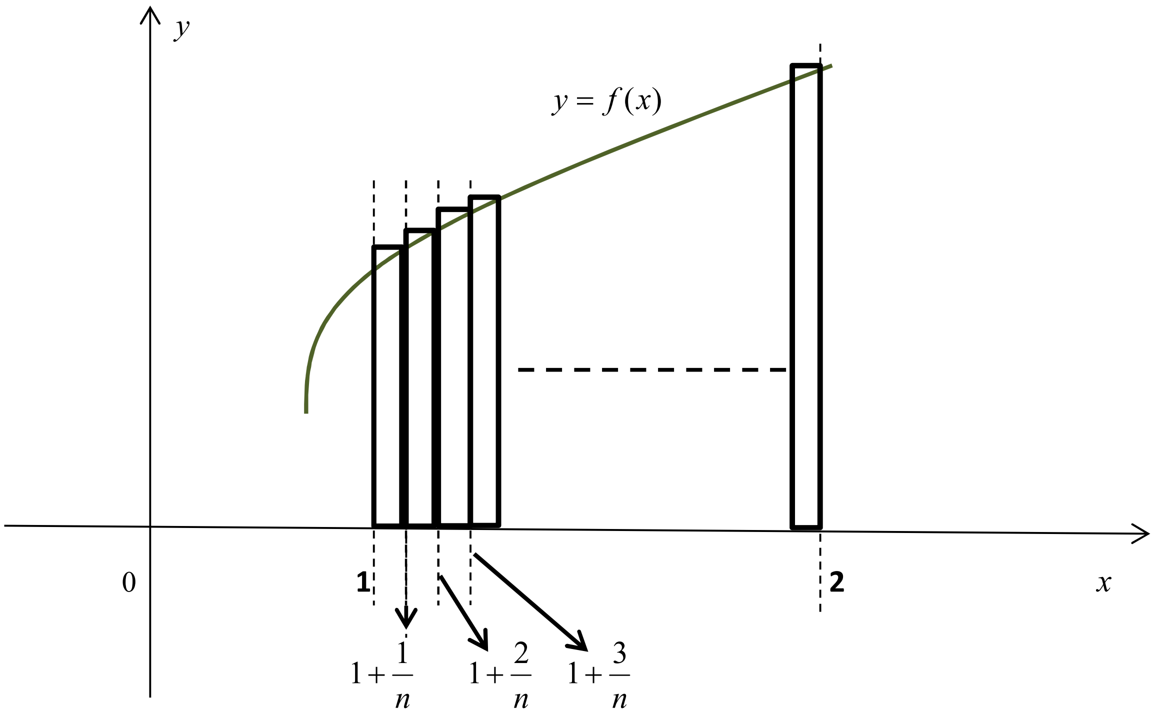
$$\frac{p^3}{27} - \frac{p^3}{9} + \frac{pq}{3} = r$$

$$\frac{pq}{3} - \frac{2p^3}{27} = r \rightarrow 9pq - 2p^3 - 27r = 0$$

$\therefore$ Necessary condition is given by $9pq - 2p^3 - 27r = 0$ (shown)

STEP III, 1999 Question 3

SOLUTIONS :



As can be observed from the above graph, total area of all rectangles for $1 \le x \le 2$

$$= \frac{1}{n} f\left(1 + \frac{1}{n}\right) + \frac{1}{n} f\left(1 + \frac{2}{n}\right) + \frac{1}{n} f\left(1 + \frac{3}{n}\right) + \ldots\ldots\ldots \frac{1}{n} f\left(1 + \frac{n}{n}\right) = \frac{1}{n}\sum_{m=1}^{n} f\left(1 + \frac{m}{n}\right)$$

When $n \to \infty$, an infinitely large number of rectangles with negligible widths are considered, expression therefore simply approaches towards the area under the graph of $y = f(x)$, ie $\int_1^2 f(x)dx$.

(shown)

$$\lim_{n\to\infty}\left\{\frac{1}{n+1}+\frac{1}{n+2}+\ldots\ldots\ldots\ldots\frac{1}{n+n}\right\} = \lim_{n\to\infty}\left\{\frac{1}{n}\left(\frac{n}{n+1}\right)+\frac{1}{n}\left(\frac{n}{n+2}\right)+\ldots\ldots\ldots\ldots\frac{1}{n}\left(\frac{n}{n+n}\right)\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{1}{n}\left(\frac{1}{1+\frac{1}{n}}\right)+\frac{1}{n}\left(\frac{1}{1+\frac{2}{n}}\right)+\ldots\ldots\ldots\ldots\frac{1}{n}\left(\frac{1}{1+\frac{n}{n}}\right)\right\} = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{m=1}^{m} f\left(\frac{1}{1+\frac{m}{n}}\right)\right\}$$

$$= \int_1^2 \frac{1}{x}dx = \left[\ln|x|\right]_1^2 = \ln 2 \ \text{(shown)}$$

$$\lim_{n\to\infty}\left\{\frac{n}{n^2+1}+\frac{n}{n^2+4}+\ldots\ldots\ldots\ldots\frac{n}{n^2+n^2}\right\} = \lim_{n\to\infty}\left\{\frac{1}{n}\left(\frac{n^2}{n^2+1}\right)+\frac{1}{n}\left(\frac{n^2}{n^2+4}\right)+\ldots\ldots\ldots\ldots\frac{1}{n}\left(\frac{n^2}{n^2+n^2}\right)\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{1}{n}\left(\frac{1}{1+\frac{1}{n^2}}\right)+\frac{1}{n}\left(\frac{1}{1+\frac{4}{n^2}}\right)+\ldots\ldots\ldots\ldots\frac{1}{n}\left(\frac{1}{1+\frac{n^2}{n^2}}\right)\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{1}{n}\left[\frac{1}{\left(1+\frac{1}{n}\right)^2-2\left(1+\frac{1}{n}\right)+2}\right]+\frac{1}{n}\left[\frac{1}{\left(1+\frac{2}{n}\right)^2-2\left(1+\frac{2}{n}\right)+2}\right]+\ldots\ldots\ldots\ldots\frac{1}{n}\left[\frac{1}{\left(1+\frac{n}{n}\right)^2-2\left(1+\frac{n}{n}\right)+2}\right]\right\}$$

$$= \int_1^2 \frac{1}{x^2-2x+2}dx = \int_1^2 \frac{1}{(x-1)^2+1}dx \text{ - - - - - - - - (1)}$$

Using the substitution $x-1=\tan\theta$, (1) becomes $\int_0^{\frac{\pi}{4}} \frac{1}{\left(\tan^2\theta+1\right)}\sec^2\theta\ d\theta = \int_0^{\frac{\pi}{4}} d\theta = \left[\theta\right]_0^{\frac{\pi}{4}} = \frac{\pi}{4}$

(shown)