The National Strategies
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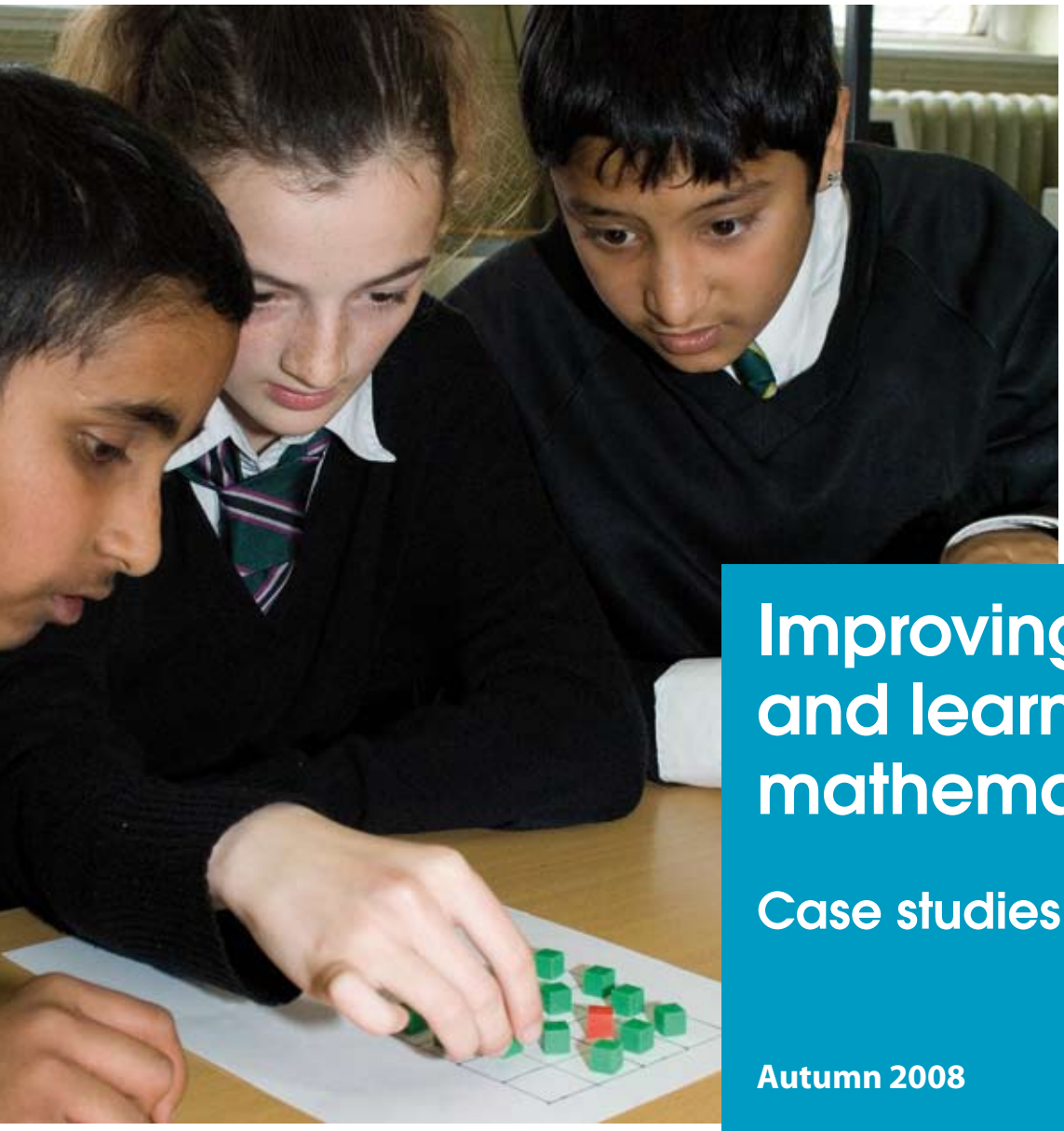

Improving teaching and learning in mathematics

Case studies

Autumn 2008

department for
children, schools and families

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Autumn 2008

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Principles into practice

Introduction

This booklet is for subject leaders and teachers of mathematics who wish to develop further their knowledge, skills and expertise in mathematics-specific pedagogy in order to improve young people's learning as they experience the new mathematics programme of study. It draws upon the following generic documents, both available as downloads from the *Framework*:

- *Pedagogy and Practice: Teaching and Learning in Secondary Schools,* DfES 0423-2004 G
- *Pedagogy and Personalisation,* DfES 00126-2007 DOM-EN

Definition (from *Pedagogy and Personalisation,* 2007):

> ***Pedagogy is the art of teaching, and the rationale that supports the actions that teachers take. It is what a teacher needs to know and the range of skills that a teacher needs to use in order to make effective teaching decisions.***

In addition, the tasks in this booklet reference and exemplify the principles outlined in *Teaching and Learning Approaches*. This is a key guidance paper, produced as part of the Strategy support for the new mathematics programme of study. It is a synthesis and interpretation of the aims, key concepts, key processes and curriculum opportunities in the new curriculum. The document can be printed from the *Framework* or the *Secondary mathematics planning toolkit disk* ('Guidance on planning' folder). It is also one of the reference papers in the booklet *Secondary mathematics guidance papers* distributed at the summer 2008 subject leader development meetings (SLDM).

The main purpose of this set of case studies and associated tasks is to provide some contextual examples of the principles outlined in *Teaching and Learning Approaches*. The studies and accompanying notes are provided as a stimulus to aid teachers' reflections on their own practice and the effects of their pedagogical decisions on pupils' learning.

You can use the tasks and case studies as a starting point to:

- stimulate a discussion about ways of working on those principles in *Teaching and Learning Approaches* that you have identified as a priority for your department;
- generate discussions about effective pedagogy in mathematics;
- review Strategy mathematics resources in order to use them to improve the teaching and learning approaches in the unit plans in your schemes of work.

Rationale for the case studies

All the case studies are a result of teachers working collaboratively to develop their pedagogical repertoire – the 'why' and 'how' of teaching decisions to promote 'deep learning' and to provide opportunities for pupils to develop key mathematical process skills. Some of them tell the story of groups of teachers working together to develop their practice in response to the challenges of the new curriculum. Some are the result of a group of teachers and consultants reflecting on how existing Strategy mathematics materials can be used to develop the key mathematical processes.

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Where to start?

You could use these tasks to support developments by:

- choosing a particular focus based on identified pupil need, then reflecting upon and adapting practice as part of an action research model;
- working with a small group of colleagues to develop a teaching and learning approach with which you are unfamiliar;
- working with the senior leader linked to your department to externally review the impact of new teaching and learning approaches.

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What is understanding in mathematics?

This task is an opportunity for individuals or groups of teachers to take time to reflect on the aims of their teaching. The discussion sets the scene for engaging with the case studies, which focus on pedagogical approaches to promote deeper understanding. Working groups of teachers and consultants have found that this simple discussion, based on reading a short article, gets to the heart of beliefs and values. These beliefs and values need to be discussed openly if colleagues are to collaborate to change classroom practice.

The task requires providing a copy to each teacher of the following paper, available on the disk accompanying this booklet:

'Relational understanding and instrumental understanding',

Richard Skemp, *Mathematics Teaching* (1976) **77**, pp. 20–26.

Teachers need to read the article in advance of meeting. They also need to have reflected on their lessons and made a few notes to inform the discussion.

Reflection task: Teaching for understanding

In preparation for this task, all involved should:

- read the Skemp article;
- note, from their lessons, one or two examples of 'relational understanding' (as defined by Skemp).

Allow **at least 15 minutes** to discuss:

- What does understanding in mathematics mean to you?
- The examples noted of a learner's behaviour indicating understanding.
- How does a teacher's behaviour shape a learner's perception of what it is to understand mathematics?

Conclude by reaching some agreement on:

- the distinction between relational and instrumental understanding;
- a few advantages/disadvantages of each;
- the relationship between teaching decisions and types of understanding.

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Exploring, applying and reviewing

There are three tasks described in this section. They can be worked on in the sequence illustrated in Figure 1 and informed by your choice of case study.

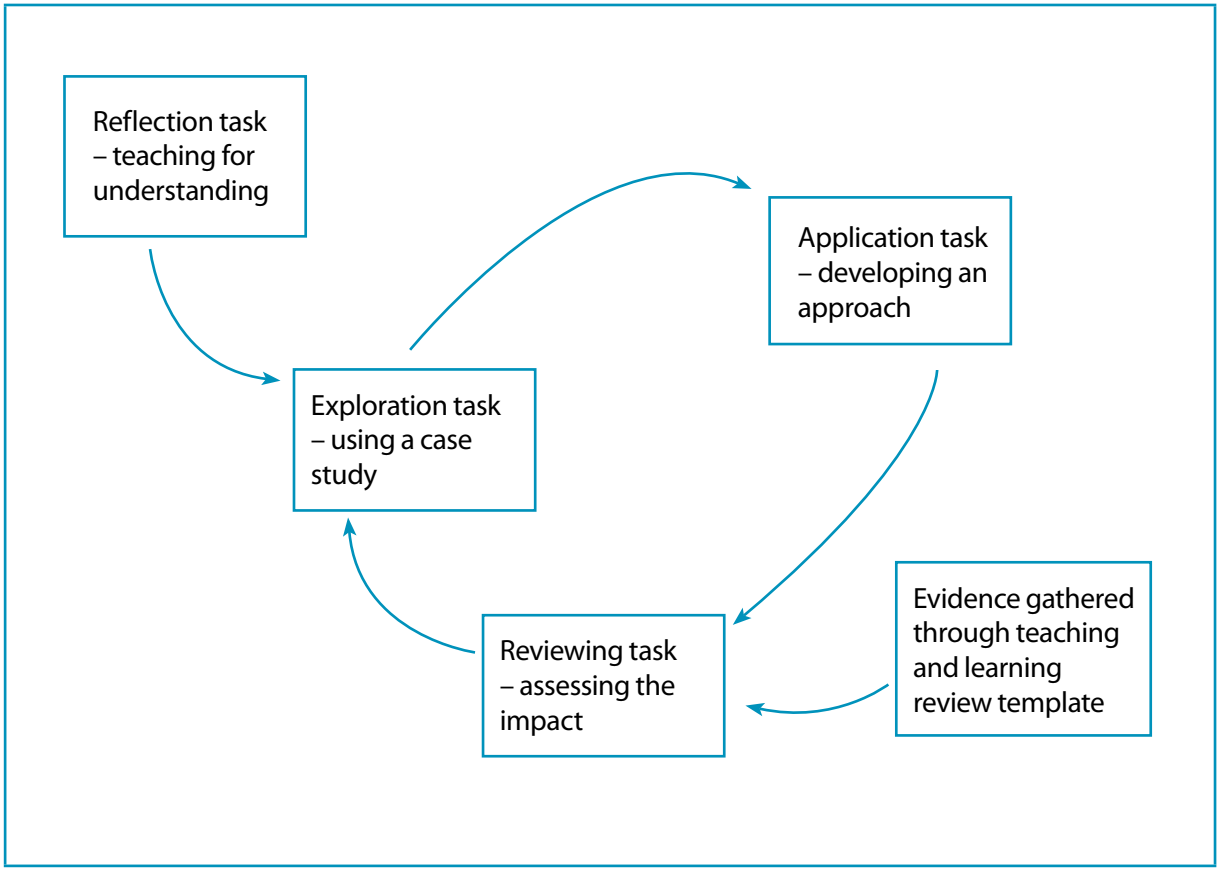

Figure 1 Developing teaching and learning approaches

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This first task requires a copy of the chosen case study for each teacher and access (one between two) to a reference copy of *Teaching and Learning Approaches*.

Exploration task: Using a case study

Choose one of the case studies to review. You may wish to read each of the short sections 'Why read this?' to help you decide which of the tasks most closely relate to your priority for development. Provide a copy of the chosen case study for each teacher. (They are available as separate files on the CD-ROM for ease of copying.)

Allow **5–10 minutes** for all teachers in the group to:

- read the case study;
- highlight any parts that they think are interesting, unusual or significant;
- think about how the actions described helped to move pupils' learning forward.

Spend **10–15 minutes** discussing the case study, being careful to explore the significance of the features that teachers have identified. Encourage them to explain:

- Why do they think the highlighted feature is significant?

Allow **5–10 minutes** to make connections between the chosen case study and *Teaching and Learning Approaches*. Point out that in each case study, the most significant teaching and learning principles addressed are highlighted on the front page. Ask pairs to re-familiarise themselves with those principles by reading the guidance paper.

Spend **10–15 minutes** using the features highlighted from the initial discussion to decide on one or two of the teaching and learning principles to consider in more depth. Ask pairs to discuss the following:

- How does the teacher use this principle?
- How does its use help the learners to make progress?
- What particular outcomes does the use of this approach promote which may have been lost had it not been used?
- Are there any pre-conditions that need to be in place in order for this principle to be effective?
- How does this principle promote deeper understanding?

Use the final **10–15 minutes** for reflection and to decide next steps:

- Which of the principles are regular features of teaching in the department?
- Which principles could be explored further?
- Are any topics or forthcoming units particularly suited to these approaches?

The next task prepares teachers to trial the teaching and learning principles which they have agreed to explore. It is a short planning exercise, which needs to have an agreed review date. It will be helpful to discuss and promote the use of the ***Teaching and learning review templates*** as a common vehicle to capture the intentions and impact of the trial. You will need, for pairs of colleagues, electronic copies of the following templates, available as spreadsheets on the *Secondary mathematics planning toolkit* disk in the 'Adaptable templates' folder

- ***Teaching and learning review template: Lessons/unit***
- ***Teaching and learning review template: Pupils' views***

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Application task: developing an approach

Remind colleagues of the main points emerging from their consideration of the selected case study. As a result of this discussion teachers selected one or two teaching and learning principles which they wish to explore further. They should now identify a target group of pupils with whom they feel comfortable trying out new ideas. Agree a date for a review meeting at which everyone will share their experiences.

It is helpful for teachers to keep a short log detailing their intentions and experiences, and the observed impact on pupils' progress and understanding because this helps to structure their feedback. Consider adapting *Teaching and learning review: Lessons/unit* to suit the development focus. Teachers need to adapt the template so that the principles they are working on are selected and copied onto the front worksheet. This can then be printed off and used as a reflection to aid review. It can be used for personal planning and reflection, as an agreed vehicle for peer planning and observation, or for observation by a senior leader.

Selecting matching prompts to gather pupils' views through small-group discussions will add to your evidence of impact. To do this you should adapt prompts on the template *Teaching and Learning review: Pupils' views*.

These templates are key to focusing on the impact of developments on the pupils' mathematical process skills.

The final stage of the sequence is to review the impact of the trialled approaches on pupils' progress and understanding, and to agree how to use the findings to further improve the quality of teaching and learning within the department's long-term development plan.

Reviewing task: assessing the impact

The aim of this task is to review the impact of the teaching and learning principles on pupils' understanding. Allow **at least 15 minutes** for pairs/small groups to share their experiences during the application task.

Ask them, in particular, to give details of:

- which principles were being developed;
- the impact on pupils' understanding and progress;
- the impact on the sequence of work;
- the impact on their expectations of their pupils.

Take feedback of significant points, keeping attention focused on the impact on pupils' understanding and progress. Ask whether using the *Teaching and learning review templates* helped to focus reflections, observations and the gathering of pupils' views.

Finally decide on the next steps needed to embed the teaching and learning principles throughout the department. You could agree to:

- work in pairs on agreed principles to share expertise and collaboratively plan some further sequences of lessons and units;
- revisit some unit plans to ensure that principles for more effective teaching and learning are explicitly referenced in the notes;
- use one or two different case studies to help you to further develop your chosen approach;
- work with your senior leader and/or colleagues from another subject area to consider how pupils can transfer and apply their learning skills across subjects;
- decide how and when to use the *Teaching and learning review templates* to help teachers to reflect on their own practice and to inform the regular review and development of the quality of teaching and learning in the department.

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Case studies

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Jump out

A small group of teachers from six schools worked together to develop pupils' functional skills in mathematics through the context of sport. The case study shows how the pupils were made explicitly aware of the mathematics involved and how this increased their motivation and helped them to see the relevance of mathematics. The pedagogical approach, teaching for deductive thinking, helped pupils to develop their mathematical reasoning skills in making and testing hypotheses.

What it did for us …

Pupil	***"I can see how maths can help athletes to perform better."***
Pupil	***"I realised that maths is in everything, even where you least expect it!"***
Teacher	***"They really enjoyed the context and could see the need to use their mathematics skill."***
Consultant	***"This is good functional mathematics. The pupils could really see the relevance of what they were doing. As well as applying existing skills they enjoyed learning some new skills (graphic calculators)."***

Why read this?

This is an example of how to develop those principles of *Teaching and Learning Approaches*[1] outlined below:

Pupils:

- learn about and learn through the key mathematical processes;
- work collaboratively and engage in mathematical talk;
- tackle relevant contexts beyond the mathematics classroom.

Teachers:

- use cooperative small group work;
- use rich collaborative tasks;
- use technology in appropriate ways.

[1] ***Teaching and Learning Approaches*** is a key document supporting the new curriculum, available from the Framework and in *Secondary mathematics guidance papers* distributed at the summer 2008 SLDM.

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Context

This joint work by a group of teachers aimed particularly to motivate pupils who usually showed limited engagement. The activities were initially planned to form part of an 'Enrichment Day' for Year 9 pupils. The teachers worked together to organise activities for the day, which was attended by pupils from six secondary schools. These activities involved the pupils making and testing hypotheses by collecting and analysing data.

The agreed outcomes for this work were to:

- develop pupils' interest and engagement with mathematics by showing them its relevance in the context of practical sporting activities;
- develop functionality by applying skills built through previous teaching;
- improve pupils' skills in working cooperatively as members of a group;
- developing teachers' understanding of teaching for deductive thinking.

Implementation

On the day, pupils took part in three different activities, one of which is described here. They worked together in mixed school groups of five or six and were judged on their performance as a group. The day contained additional, sport-related, individual challenges and lots of prizes.

> What questions would you ask to probe pupils' understanding of statistical variables and correlation?

The pupils were initially asked to explore the effects of varying values of their choice on performance in the long jump. They were given some ideas for variables that would be feasible to test and most chose length or speed of run-up.

> The deductive model involves pupils testing initial hypotheses by analysing the data to confirm or refute. In which other areas of the mathematics curriculum do you develop this type of thinking?

The model was 'deductive' in that pupils formed an initial hypothesis and then tested it in the field by collecting primary data at the long jump pit.

> Is it more helpful to plan some pre-work on the use of graphic calculators as portable devices to analyse the mathematics outside the classroom?

All the groups shared graphic calculators. The teachers helped the pupils to use them to quickly plot scatter diagrams, look for patterns and analyse correlation. For many this was their first experience of graphic calculators and so this formed part of their learning.

The use of graphic calculators sped up the analysis, freeing time for the pupils to refine their ideas and retest. Through group discussion of their results they were quickly able to relate their findings to their original hypothesis, identifying whether the data supported or refuted their conjecture. Their discussion included consideration of sample size and the effects of other variables.

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Impact on learners

The pupils were very motivated by the link to sport, and by the opportunities to work together on a 'real' task. They saw the need to apply their mathematical skills, and this engaged pupils who did not usually enjoy mathematics.

The group discussions strengthened their investigative and reasoning skills and helped them to develop an understanding of the extent to which data can be used to support causal relationships.

> Think about how the teachers would have used 'mini-plenaries' throughout to develop pupils' reasoning skills.

The opportunity for pupils to work both outside the subject area and outside the physical classroom area required them to transfer those skills already taught in their mathematics lessons into a real-life context; this transfer helped them to develop towards mastery of those skills.

> Which other curriculum areas could provide fruitful contexts in which to develop pupils' functional skills?

Impact on teachers

The teachers saw how the opportunity to work outside the classroom motivated pupils. They were impressed by the quick and easy way that using graphic calculators could enable pupils to focus on the interpretation of data at the point of collection. They developed their understanding of teaching for functional skills by discussing their observations of the pupils' work. They began to appreciate the nature of the teaching approaches required to engage their pupils in mathematical deductive thinking, and were keen to test their understanding with colleagues in other disciplines.

Next steps

The teachers involved plan to make more explicit links between different subject areas in their teaching of mathematics. In particular they will look at where key processes can be developed in contexts outside the mathematics classroom.

Further references

If you would like to know more about different teaching models or about teaching and learning functional mathematics, you may like to read:

- *Pedagogy and Practice*, Unit 2: Teaching models, *Secondary mathematics planning toolkit,* 'Pedagogy' folder
- *Teaching and learning functional mathematics: Resources to support the pilot of functional skills, Secondary mathematics planning toolkit,* 'Rich tasks' folder.

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Consecutive … or not?

A small group of teachers from different schools worked together to develop and trial a rich collaborative task. The aim was to improve pupils' confidence and motivation to learn mathematics. The case study illustrates a quick and adaptable technique that engages pupils in class discussion in order to extend their use of mathematical language. The pedagogical approach enabled pupils to develop an understanding of a mathematical concept by considering positive and negative examples of it.

What it did for us …

Pupil	***"I liked being able to explain my ideas. I didn't expect everyone to want to know what I thought."***
Teacher	***"I never thought these pupils could engage in this way. They can probably do more thinking than I expected."***
Teacher	***"At first some pupils didn't seem sure what was going on. The activity meant they had to listen to other people's ideas and come up with some of their own. The more they listened and spoke, the more they contributed. It was like they were solving a little mystery, together… and they did it, together… This was the best classroom atmosphere we have had so far this year."***

Why read this?

This is an example of how to develop those principles of *Teaching and Learning Approaches*[1] outlined below:

Pupils:

- learn about and learn through the key mathematical processes;
- work on sequences of tasks;
- select the mathematics to use;
- work collaboratively and engage in mathematical talk.

Teachers:

- use cooperative small group work;
- use rich collaborative tasks;
- develop effective questioning.

[1] ***Teaching and Learning Approaches*** is a key document supporting the new curriculum, available from the Framework and in *Secondary mathematics guidance papers* distributed at the summer 2008 SLDM.

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Context

This joint work explored the effectiveness of the teaching technique 'Concept attainment' for developing pupils' understanding. The technique, commonly known as 'I like / I don't like' or 'Yes / No', asks pupils to decide which, of the examples they are given, meet the conditions of the concept.

The working group involved teachers from three different schools; two of the teachers were new to teaching mathematics. All were concerned about the progress of small groups of pupils who lacked confidence and were experiencing difficulties in mathematics.

The activities were developed and trialled with lower sets in Years 7, 8 and 9. Initial observations of the classes showed that the pupils struggled to form their own explanations of situations, and needed opportunities to talk about their mathematics and to develop their use of vocabulary. These pupils had difficulty in understanding new ideas and were not confident with some basic number skills such as adding two two-digit numbers. Many of the pupils displayed poor attitudes to learning, and limited engagement in mathematical activity. Some poor behaviour was also observed.

The agreed outcomes for this work were to:

- try a technique designed to help pupils understand a new concept;
- give pupils opportunities to explain their findings, and to handle mathematical vocabulary in their explanations;
- develop pupils' number skills;
- engage pupils in mathematical activity through investigative work;
- give teachers a better understanding of the pedagogical approach of 'teaching for concept attainment'.

Implementation

It was agreed that exploring the task 'Consecutive sums' would provide an opportunity for pupils to demonstrate their skills, or reveal their difficulties, with adding together pairs of numbers, and to help them explain their own findings.

Before beginning the investigation a starter was used to help the pupils establish and understand the meaning of the word 'consecutive'. First, it was explained that during the lesson pupils would need to choose examples of consecutive numbers. The teacher then wrote 'Consecutive or not?' on the board, and placed a pair of consecutive items in a column headed 'Yes', and a pair that were not consecutive in a column headed 'No'. Pupils were asked to contribute an example to one of the columns, beginning with a positive example.

Consecutive or not?	
Yes	**No**
January, February 2, 3 Sunday, Monday, Tuesday	Monday, Wednesday, 2, 5 February, October, December

It took at least five minutes before pupils were confident to offer their own examples; during this time the teachers offered a few more examples and encouraged pupils to 'have a go'. As soon as one or two had started to contribute, others quickly joined in. The whole class discussed and reached a decision about each example. At this stage the pupils were engaged in trying to find the meaning of the word consecutive.

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Finally, when decisions were being made more easily, pupils were asked to explain, in their own words, what 'consecutive' meant. The different explanations were evaluated by the class and judged according to how well these words described the concept. At this stage pupils were trying to communicate their emerging understanding to others.

> Does the struggle to reach a definition help pupils to understand and remember the term?

The pupils were then in a position to move on to investigate 'Consecutive sums'. This investigation invited pupils to choose any pair of consecutive numbers and add them, then to repeat this and see what they noticed. It gave opportunities for pupils to work systematically, to tabulate results, to search for patterns or rules, to generalise and to express this generality using symbols.

Probing questions were focused on the pupils' selection of the mathematics to use, for example:

- Why did you choose to start with these numbers?
- Where is an easy place to start?
- How could you present your results?

> These types of questions are useful to make explicit the decision-making element of the process skills.

Impact on learners

In every lesson pupils fully engaged with the starter activity. They liked the idea of a mystery to solve (finding the meaning of 'consecutive'), so despite being challenged the classes still wanted to succeed. It was crucial for pupils to provide their own examples and they were happy when these proved to be correct. When examples were incorrectly placed, the group helped to modify the thinking in what became a supportive learning environment.

Cooperative small–group work began to emerge when pupils collaborated in pairs to agree a definition of 'consecutive'. The definitions were then shared and evaluated by the whole class. At the end of this episode their explanations and arguments revealed a sound conceptual understanding of the meaning of 'consecutive'.

> The activity was simple to access and required group talk in order to develop and refine explanations.

Pupils engaged in the investigative task immediately. There was generally a good understanding of what they were being asked to do and pupils were happy to continue working together. The small groups worked effectively with pupils sharing ideas and approaches and checking one another's results.

The pupils began to explain what they had found, and were motivated by having their views and findings verified and acknowledged. They developed increasing confidence with their number skills and in their use of mathematical vocabulary as they recognised that they were having success with strategies and results.

> Think about how the teachers would have used 'mini-plenaries' to develop pupils' mathematical vocabulary and reasoning skills.

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Impact on teachers

Teachers were surprised and encouraged by seeing the pupils engaged, and began to widen their view of how and what pupils should be learning. They moved away from talking about these groups in a negative way. Their conversations focused less on the skills that pupils couldn't perform or facts that they couldn't remember. Instead, they began to consider the importance of activities that enabled pupils to think and talk mathematically. They all agreed on the need to develop explicitly the key processes in mathematics, especially ways of solving problems. They felt that planning for questioning about the processes was a crucial part of making this work in the classroom.

Next steps

The teachers continued the work back in their schools. One began to develop a scheme of work for their Year 7, using a range of activities that focused on developing pupils' use of language and on investigative tasks. Another teacher contributed to departmental collaborative planning, sharing ideas of engaging activities when planning units of work.

The joint work continued to explore the technique for developing *Concept attainment*, and many other possibilities were identified.

Further references

If you would like to know more about different teaching models, you may like to read:

- *Pedagogy and Practice*, Unit 2: Teaching models, *Secondary mathematics planning toolkit,* 'Pedagogy' folder.

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New handles for old angles

In this study a department worked together to develop a unit of work in geometry. Feedback from pupils, as part of a school behaviour review, had convinced the teachers of the need to increase pupil engagement. They decided to do this through cooperative small–group work, using tasks which required pupils to make decisions about the mathematics to use. The pedagogical teaching model featuring strongly in the unit is 'teaching for inductive thinking'.

What it did for us …

Pupil	***"I liked working in groups, and marking my partner's work. It helped me to see how to improve my answers."***
Teacher	***"I became much more confident in planning and managing group discussions. The pupils' thinking really developed through articulating their reasoning."***

Why read this?

This is an example of how to develop those principles of *Teaching and Learning Approaches*[1] outlined below:

Pupils:

- learn about and learn through the key mathematical processes;
- work collaboratively and engage in mathematical talk;
- work on sequences of tasks.

Teachers:

- develop effective questioning;
- expose and discuss misconceptions;
- use cooperative small group work;
- use rich collaborative tasks;
- use technology in appropriate ways.

[1] ***Teaching and Learning Approaches*** is a key document supporting the new curriculum, available from the Framework and in *Secondary mathematics guidance papers* distributed at the summer 2008 SLDM.

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Context

As part of a behaviour review, pupils were asked about their perceptions of mathematics, their attitudes to the subject and their experiences in lessons.

After discussing the findings, the mathematics department decided to work collaboratively to develop a unit of work which would better engage the pupils in their learning. They decided to focus on a key group in Year 9, all of whom had a target of level 5. Geometry was selected as a priority area because this often emerged as a weakness from Key Stage 3 and GCSE test question analysis.

The agreed outcomes for this work were to:

- improve the engagement of pupils in mathematics;
- improve teachers' skills in unit planning to include key principles from *Teaching and Learning Approaches;*
- incorporate rich tasks from Strategy and Standards Unit materials.

Implementation

The teachers in the department worked collaboratively to develop a unit based on the objectives from the first 'Geometry and measures' unit in Year 9. The lesson sequences were trialled with the target group and the department then reviewed the lessons, made changes and adapted the unit plan.

> Do your pupils know when you are trying out something new?

The pupils were clearly informed about what the unit was attempting to address and they were asked their opinions on the lessons.

In planning the unit the teachers selected strategies so that they were helping pupils to learn through the key mathematical processes. They did this by making aspects of the processes explicit through activities such as sorting and classifying shapes on a Venn diagram. The pupils worked on sequences of tasks designed to make connections between visual representations, shape properties and logical proof. In one sequence the pupils experienced a range of activities such as visualising and classifying shapes, 'odd one out' and 'true/false'. All of these involved the pupils in exposing their understanding, helping the teachers to assess prior learning and providing the opportunity to discuss misconceptions.

> The inductive model requires pupils to sort, classify and re-sort data to begin to make hypotheses that can be tested in future work.
>
> Can you think of other areas of the mathematics curriculum which provide opportunities to develop inductive thinking?

The next step was to prove the size of angles inside a triangle and angles inside a polygon. At this point teachers were trying to develop inductive thinking, so that pupils worked logically from given facts to derive and prove properties. They used overlays of lines and shapes, each representing a known or given fact, to build up a geometrical proof. Reasoning towards the proof was worked on by the whole class and then reconstructed in pairs and groups. At each stage this involved pupils working cooperatively and engaging in mathematical talk.

Resources were drawn from *Interacting with mathematics at Key Stage 3: Year 9 geometrical reasoning mini pack*. The unit also drew on activities and strategies from the *Leading in Learning* (LiL) handbook particularly using a 'mystery' and 'collective memory'.

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In the concluding phase of the unit the pupils worked in pairs on specific Key Stage 3 test questions. They worked individually on different questions, then marked one another's answers using a provided mark scheme. This allowed for a rich discussion of the adequacy or otherwise of answers, particularly to questions of the type: 'Explain how you know'.

Could you use past paper questions differently, so that they engage pupils in mathematical discussion?

Impact on learners

The pupils enjoyed the chance to work in pairs or groups and the opportunities to discuss mathematics. They became more confident in explaining their reasoning and more able to question each other as the unit progressed. They gave very positive feedback on the lessons and the teacher reported that their behaviour and engagement improved.

Impact on teachers

The teacher of the trial group had expressed initial concern that some of the activities were 'risky'. She was particularly concerned over the visualisations and the 'collective memory' exercises. As the unit developed she was surprised and pleased to see the level of engagement from the pupils. She also became more confident in conducting constructive classroom discussion. During the teaching sequences she found that she was allowing the pupils to learn more for themselves and from each other rather than leading from the front. She realised that, in trying a new approach for the first time, the careful selection and planning of activities is crucial. In addition, the teacher felt reinvigorated by the experience. She felt her teaching in general had improved and her own enjoyment and engagement had increased.

In discussion the department talked of the need to support each other and to reflect together on the success or otherwise of new approaches. It was also decided that the pupils needed to be aware of the change in approach and made to feel that they are involved in their teachers' own learning experiences in developing their teaching skills.

Next steps

The department now intends to focus on developing the follow-up 'geometry and measures' units in Year 9. This could include:

- pupils learning about the relevance of mathematics through work on wallpaper designs. This could be on paper initially and then refined and developed using transformations involving dynamic geometry software;
- pupils looking at work on tessellations by M. C. Escher so that they appreciate the historical and cultural roots of mathematics.

They are also looking at other areas of mathematics for opportunities to develop thinking skills by using teaching strategies drawn from the Leading in Learning (LiL) handbook.

Further references

If you would like to know more about different teaching models, sources of ideas for teaching geometry or more ideas on teaching strategies for developing thinking skills, you may like to read:

- *Pedagogy and Practice*, Unit 2: Teaching models, *Secondary mathematics planning toolkit,* 'Pedagogy' folder
- *Interacting with mathematics in Key Stage 3: Year 9 geometrical reasoning, Secondary mathematics planning toolkit,* 'Rich tasks' folder
- *Leading in Learning; Developing thinking skills in secondary schools*, *Secondary mathematics planning toolkit,* 'Rich tasks' folder.

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Clouding the picture

A small group of teachers and consultants worked with this familiar Key Stage 3 resource to illustrate how it could be used in the context of the new programme of study. The activity described provides opportunities for pupils to develop their process skills in algebra. In particular it allows them to make connections between equivalent forms of equations and to develop more flexible strategies for solving equations. The group decided that the prevalent pedagogical approach was teaching for constructing meaning and the use of cognitive conflict.

What it did for us …

Teacher	***"This activity is really powerful in helping them to develop their confidence and skills in manipulating equations."***
Consultant	***"It made me realise that I need to get some networks of teachers working on some of our 'old favourites' so that they see the potential of such tasks to embrace the new curriculum."***
Consultant	***"I had forgotten this resource and must make sure more departments write it into their schemes of work – it's so good."***

Why read this?

This is an example of how to develop those principles of *Teaching and Learning Approaches*[1] outlined below:

Pupils:

- work collaboratively and engage in mathematical talk;
- work on sequences of tasks.

Teachers:

- build on the knowledge pupils bring to a sequence of lessons;
- expose and discuss misconceptions;
- use cooperative small group work;
- use rich collaborative tasks.

[1] ***Teaching and Learning Approaches*** is a key document supporting the new curriculum, available from the Framework and in *Secondary mathematics guidance papers* distributed at the summer 2008 SLDM.

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Context

'Clouding the picture' is an activity that focuses on creatively transforming algebraic expressions and equations. It develops skills of algebraic manipulation and also allows pupils to make links between different forms of equations. The resource is described in *Teaching mental mathematics from level 5: Algebra* (available on the *Secondary mathematics planning toolkit disk*, in the 'Rich tasks' folder). It was also filmed as an activity in two classrooms and distributed as part of the autumn 2005 SLDM.

The outcomes that have been observed from this activity are that:

- pupils develop techniques that they understand in order to transform and solve equations;
- pupils self-assess knowledge and skills in order to prioritise their learning;
- teachers focus on enabling pupils to create connections;
- teachers have a better understanding of the role of 'cognitive conflict' in promoting pupils' learning.

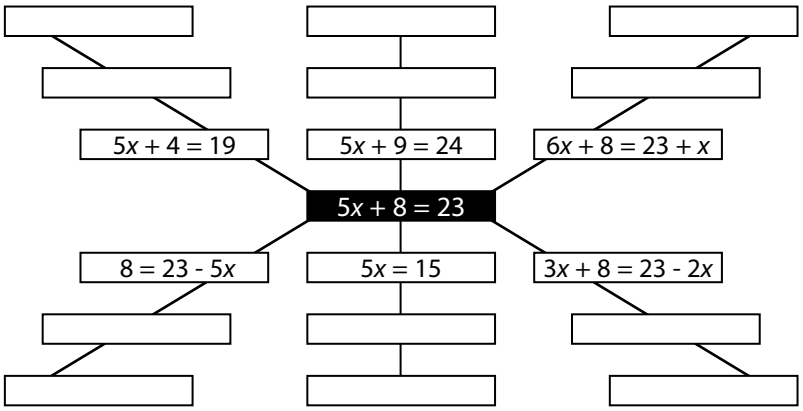
Implementation

> How do you allow pupils to display their prior knowledge of a topic?
>
> How does that information inform your response?

The sequence begins by presenting pupils with a set of equations and asking them to work in pairs or small groups to classify the equations into those which they can or cannot easily solve. This helps pupils to become explicitly aware of their learning needs, and helps teachers to assess the prior knowledge that pupils bring to the task.

Pupils then engage with a rich collaborative task that is intended to provoke cognitive conflict. This is done by providing a single equation at the centre of a web diagram from which pupils follow self-generated rules to complete arms of the web with equivalent equations.

Clouding the picture: algebra 2



The teacher acts as a facilitator, allowing pupils to build on existing algebraic skills to construct meaning about the equivalence of different algebraic equations. Pupils soon realise that seemingly complex equations are equivalent to much simpler forms. They then see the purpose of transforming algebraic equations into equivalent forms that can be more easily solved.

> What is the teacher's role in this episode of the lesson?

The activity is constructed so that pupils are encouraged to work collaboratively and engage in mathematical talk. Pupils have the opportunity to discuss common misconceptions as they agree on the transformations of the equations.

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Pupils are then directed back to their original 'can/cannot solve' classification of equations. Cognitive conflict occurs when pupils recognise equivalence between forms they have classified as both 'can solve' and 'cannot solve'. The activity concludes with pupils solving an unrelated equation that has an unknown on both sides.

> Can you think of another lesson which you have taught recently that provoked 'cognitive conflict'? What was the effect on pupil learning?

Impact on learners

Pupils are guided through a stage of cognitive conflict towards making connections between different representations of the same equation. This is an example of a 'constructing meaning' approach, where the teacher 'scaffolds' the learning enabling pupils to construct their own conceptual understanding.

They are able to build confidence in manipulating equations and make connections in their understanding by forming 'chains' such as this one below:

$$\begin{aligned} 3x + 3 &= 7 + x \\ 2x + 3 &= 7 \\ x + 3 &= 7 - x \\ 3 &= 7 - 2x \\ 3 - x &= 7 - 3x \end{aligned}$$

Pupils also learn to connect $2x + 3 = 7$ with $0.2x + 0.3 = 0.7$ and $0.02x + 0.03 = 0.07$.

As pupils explain their chains and connections to each other their understanding deepens. Their confidence with algebra grows as they recognise that they are having success by working logically and systematically. From this point 'unknown on both sides' is no longer seen as too hard to solve.

> Are there other topics for which a 'clouding the picture' activity would be useful?

Impact on teachers

Teachers are made aware of pupils' prior learning as they can see which equations pupils classify as 'difficult'. As they listen to the pupils' discussions during group work, they become more aware of the conceptual chains of reasoning required for pupils to confidently transform equations and to recognise equivalences. They gain a better understanding of the role that cognitive conflict can play in the learning process, and see the benefits to pupils' understanding when they make their own connections rather than learning discrete, teacher-given techniques.

Next steps

The scaffolding of the web diagram can be dismantled as pupils develop their thinking. Eventually pupils can be guided to work without the diagram to transform the particular equation to one that they can solve, emphasising the method that they are using rather than the answer.

Further references

If you would like to know more about different teaching models and sources of ideas for teaching algebra, you may like to read:

- *Pedagogy and Practice*, Unit 2: Teaching models, *Secondary mathematics planning toolkit,* 'Pedagogy' folder
- *Interacting with mathematics in Key Stage 3: algebraic reasoning, Secondary mathematics planning toolkit,* 'Rich tasks' folder
- *Teaching mental mathematics from level 5 algebra*, *Secondary mathematics planning toolkit,* 'Rich tasks' folder.

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The proof of the pudding (is in the teaching!)

This case study describes collaborative work by teachers in two different schools working with a local authority (LA) consultant to develop pupils' skills of reasoning and proof. In both schools geometrical reasoning was identified as an area needing further development; the teachers were motivated to develop more effective teaching methods to promote this strand of mathematical thinking. They selected teaching for inductive thinking and deductive thinking as an effective pedagogical approach.

What it did for us …

Pupil	***"I need to use what I've done before and try to connect it with what I'm doing now so I don't have to remember more things, otherwise I get a headache!"***
Teacher	***"The pupils loved matching circle diagrams with their theorems. It helped them to see what the theorems look like by using a visual and kinaesthetic-type activity."***
Consultant	***"Learning mathematics ceases to be hard work once they enjoy it. Pupils can capture the 'buzz' for learning through a variety of pedagogical approaches."***

Why read this?

This is an example of how to develop those principles of *Teaching and Learning Approaches*[1] outlined below:

Pupils:

- work collaboratively and engage in mathematical talk;
- work on sequences of tasks.

Teachers:

- develop effective questioning;
- emphasise methods rather than answers;
- use rich collaborative tasks.

[1] ***Teaching and Learning Approaches*** is a key document supporting the new curriculum, available from the Framework and in *Secondary mathematics guidance papers* distributed at the summer 2008 SLDM.

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Context

The teachers in each department worked collaboratively to plan and trial some new approaches to circle geometry involving the use of specifically designed pupil answer boards or working sheets. Pupils and teachers completed feedback forms to help evaluate impact on learning.

In one school the target group comprised Year 11 pupils working on the C/D borderline who were already familiar with the content being addressed. In the other school the target group was a Year 8 higher-ability group for whom the content was totally new.

The agreed outcomes for this work were to:

- develop pupils' skills in using mathematical vocabulary to explain their findings;
- develop pupils' geometric reasoning skills and their understanding of proof;
- develop teachers' understanding of teaching for inductive and deductive thinking.

Implementation

The teachers in both schools planned a sequence of lessons around circle geometry focusing on the process of proof. The teachers were also working with LA consultants to explore different pedagogical models.

> How does dynamic geometry software or the dynamic use of images support the understanding of proof?

The initial episode of the sequence exemplified teaching for inductive thinking. The Year 8 group explored circle diagrams with their teacher using the interactive whiteboard (IWB). The Year 11 group used a dynamic geometry software package to generate angle value data. Pupils in both groups were asked to use the results of their explorations to formulate an initial hypothesis relating the angle subtended by an arc at the centre of the circle to the angle subtended by the same arc at the circumference.

The second episode of the sequence focused on the development of reasoning through deductive thinking. Using known properties of isosceles triangles, pupils were encouraged to 'deduce' unknown angle sizes in geometric diagrams. The teachers then used carefully planned questions to move pupils towards using algebra as a method of generalisation. Throughout this episode the pupils worked collaboratively to discuss and develop the language of explanation and generalisation. Teachers used 'mini-plenaries' to support and develop pupils' use of the language of generalisation towards an understanding of proof.

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Probing questions focused on the pupils' deductive thinking, for example:

- How do you know that?
- Is that sometimes/ always/ never true?
- Can you explain that in your own words?
- Can you explain that on paper for someone else to understand?

Does your planning involve the use of probing questions?

Do you discuss this as a department?

The lesson aims for the two groups then diverged:

In the Year 8 group, pupils were introduced to the properties of cyclic quadrilaterals. In pairs they were asked to generalise about angles in a cyclic quadrilateral using the geometric reasoning skills they had developed in the earlier episode. They communicated and reflected by engaging in mathematical discussion.

The teachers said: 'you are mathematicians looking for answers.' This proved a powerful motivator.

How can you encourage pupils to see themselves as mathematicians?

In the Year 11 group the aim was now to develop a formal representation of the proof. The pupils worked in small groups to place cards with algebraic components of a proof in the correct order. They listened to and discussed the precision of one another's mathematical reasoning.

How important is it for learners to talk and work collaboratively to succeed with more formal written proofs?

Teachers of both groups used a recording sheet to evaluate the observed impact on learning. This was further evaluated through teacher feedback on the strategies used. In both cases this discussion itself developed the teachers' awareness of different teaching approaches and their impact.

With both groups the key processes most significantly developed were:

- working logically;
- reasoning deductively;
- engaging with someone else's mathematics;
- differentiating between evidence and proof;
- reflecting using thinking and reasoning;
- communicating using precise language and symbolism.

Think about how the teachers would have used 'mini-plenaries' throughout this sequence of lessons to develop reasoning skills.

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Impact on learners

Year 8

The pupils could see how they could use reasoning to solve geometrical problems. They appreciated the difference between finding a solution in a particular instance and producing a general argument or proof. They felt more confident to evaluate someone else's explanation.

Year 11

The pupils' evaluations showed that they particularly enjoyed opportunities to collaborate on group presentations of problems to their peers. They recognised that discussing, listening and arguing helped them to become more confident and independent learners. They were prepared to express partially-formed thinking and to extend their learning in groups through dialogue generated by the use of dynamic geometry software and probing questions. They focused on methods rather than answers when working on their problem and, as a result, they were developing their own chain of reasoning and making connections between the new learning and prior knowledge.

Impact on teachers

The teachers of the Year 8 class identified pedagogies which impacted positively on pupils' learning, particularly the way the planned IWB resource had been used to scaffold the deductive reasoning processes. They reflected on their own practice and made suggestions for improvement. The feedback contributed to their professional development because the lesson sequences provided a shared experience of teaching approaches and observation of impact on pupils' learning.

The Year 11 teachers gained a better understanding of the importance of questioning in creating effective dialogue in their classroom. They agreed, in consequence, to overtly plan probing questions as they developed other units of work.

Next steps

The approaches described could be further developed by working in other units on geometrical reasoning, highlighting key processes and the principles for effective teaching and learning.

The consultants hope to extend the use of dynamic geometry software to generate class discussion on demonstration and proof in other departments. They are making sure all departments have access to GeoGebra, a free dynamic mathematics software application which joins geometry, algebra and calculus. GeoGebra is available at **www.geogebra.org**

Further references

If you would like to know more about different teaching models, sources of ideas for teaching geometry or more ideas on teaching strategies for developing thinking skills, you may like to read:

- *Pedagogy and Practice,* Unit 2: Teaching models, *Secondary mathematics planning toolkit,* 'Pedagogy' folder
- *Interacting with mathematics in Key Stage 3: geometrical reasoning*, *Secondary mathematics planning toolkit*, 'Rich tasks' folder
- *Leading in Learning*; *Developing thinking skills in secondary schools*, *Secondary mathematics planning toolkit*, 'Rich tasks' folder.

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Wise words

A small group of teachers and consultants worked with this familiar Key Stage 3 resource to illustrate how it could be used in the context of the new programme of study. The activity described provides opportunities for pupils to develop their process skills in statistics, in particular their skills in interpreting data. The group decided that the underlying pedagogical approach was 'teaching to enable pupils to construct meaning'.

What it did for us …

Pupil	***"I liked using the whiteboards to keep redrafting. It's the way we do it sometimes in English."***
Teacher	***"They have become much more confident in using technical language."***
Consultant	***"Some teachers were unaware of these resources. They were really pleased to see that they address the teaching of mathematical process skills."***

Why read this?

This is an example of how to develop those principles of *Teaching and Learning Approaches*[1] outlined below:

Pupils:

- work collaboratively and engage in mathematical talk;
- select the mathematics to use;
- work on sequences of tasks.

Teachers:

- build on the knowledge pupils bring to a sequence of lessons;
- expose and discuss misconceptions;
- use cooperative small group work;
- use rich collaborative tasks.

[1] ***Teaching and Learning Approaches*** is a key document supporting the new curriculum, available from the Framework and in *Secondary mathematics guidance papers* distributed at the summer 2008 SLDM.

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Context

"Wise words" is a versatile task suitable for developing understanding of most visual forms. Pupils work in pairs, with identical sets of up to eight cards. Pairs compose two statements to describe a chosen card and their opposing pair must try to identify the card, from the statements. The two statements should focus on different key features and use accurate vocabulary. The choice of object or image on the cards, the number of items and the key words make this a rich and adaptable activity, engaging pupils in discussion and forcing them to consider the precision of the language they are using.

Used in the context of statistics, this activity helps to develop pupils' ability to interpret and describe data sets using correct and precise mathematical language. The structure of the materials supports teachers in encouraging mathematical dialogue. The resource was developed as part of *Interacting with mathematics in Key Stage 3: Handling data* materials and is available on the *Secondary mathematics planning toolkit disk*, in the 'Rich tasks' folder.

> How good are your pupils at using technical language to describe mathematical situations?

The outcomes that have been observed from this activity are that:

- pupils are motivated by the competitive element of the activity to accurately recognise similarities and differences between different representations of data;
- they improve their skills in interpreting data represented graphically;
- they work collaboratively and are forced to engage in using accurate mathematical vocabulary;
- teachers develop their skills in planning for, and managing, classroom dialogue to promote pupils' learning.

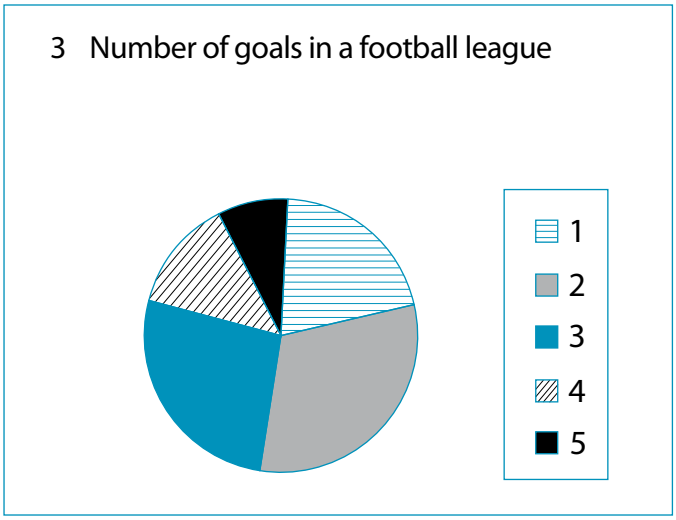
Implementation

Pupils worked in pairs using mini whiteboards to draft statements about charts using specific given key words. For example, the instruction could be:

'Choose a chart and make one statement about totals or proportion and one statement about mode or range, using some of these key words:

mode, range, total, proportion, percentage, decimal, fraction.'

Example of chart



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The opposing pair of pupils worked to identify the chart from the set of charts and graphs. The groups of four were then encouraged to critique and revise one another's statements until they reached a group consensus on an accurate description.

The groups were then asked to discuss which data sets were more difficult than others to identify or describe, and the benefits of using some key words over others.

> How well does your current teaching of statistics exploit the potential of collaborative group work?

Impact on learners

This rich collaborative task allowed pupils to work inductively classifying data sets in order to describe them using appropriate technical language.

> The inductive model requires pupils to sort, classify and re-sort data to begin to make hypotheses that can be tested in future work. Can you think of other areas of the mathematics curriculum which provide opportunities to develop inductive thinking?

The accessible and open nature of the first episode (creating an initial statement) allowed teachers good opportunities to assess prior knowledge. For example, it quickly became clear whether or not pupils understood the key vocabulary.

The second episode in which pairs reviewed and critiqued each other's statements promoted cognitive conflict when pupils realised that some key words cannot be used to describe some types of data. This also provided the teacher with an opportunity to discuss misconceptions about the graphical representation and the language we use to interpret and describe it.

> How can key words be used in other contexts to help scaffold dialogue and model correct mathematical language?

This peer discussion helped them to construct their own meaning of the key words and hence reach a deeper understanding of why some charts are more suitable for some types of data set.

> How does peer discussion help pupils to construct meaning?

Throughout the task, pupils were encouraged to engage in mathematical discussion of results in a highly structured way; this supported their learning and led to more precise written statements. The nature of the task enabled pupils to engage with someone else's mathematical reasoning as they worked to form convincing arguments. They took more risks because they liked the fact that the mini whiteboards allowed them to revise and rewrite their statements as often as necessary.

> Could you apply the same template of 'draft, review, revise' to other topics?

Impact on teachers

Teachers managed and structured group discussion using the guidance given for the task. This helped them to appreciate the effect of a carefully structured approach and they have extracted and adapted this approach for use in other contexts.

Next steps

The teachers were keen to find other resources to develop pupils' skills in interpreting and explaining data through mathematical discussion and written explanations. They decided to explore the suite of 'Handling data materials', available through the *Interacting with mathematics at Key Stage 3* resources. They realised they would need to consider other pedagogical models suited to different activities.

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Further references

If you would like to know more about different teaching models or sources of ideas for teaching statistics through the key processes, you may like to read:

- *Pedagogy and Practice*, Unit 2: Teaching models, *Secondary mathematics planning toolkit,* 'Pedagogy' folder
- *Interacting with mathematics in Key Stage 3: handling data, Secondary mathematics planning toolkit,* 'Rich tasks' folder.

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Are you sure?

This study describes how two newly-qualified teachers (NQTs) worked together to develop their practice in teaching to develop thinking skills. The focus of the work was on planning to provoke cognitive conflict. The teachers were working with the LA consultant to explore the benefits to learners of the pedagogical approach described as 'constructivist'.

What it did for us …

Pupil	***"I liked being able to argue and get my thoughts clear."***
Teacher	***"I was worried that this approach would unsettle them, but they can see the benefit now. It was really helpful to watch my colleague teach the same lesson to her class."***
Teacher	***"I've become much better at dealing with misconceptions. I now use them as great teaching opportunities."***

Why read this?

This is an example of how to develop those principles of ***Teaching and Learning Approaches***[1] outlined below:

Pupils:

- work on sequences of tasks;
- work collaboratively and engage in mathematical talk.

Teachers:

- build on the knowledge pupils bring to a sequence of lessons;
- expose and discuss common misconceptions;
- develop effective questioning.

Context

The two NQTs were working in a department where approaches to thinking skills and cognitive acceleration were well established. They were keen to develop their practice in planning for cognitive conflict as they recognised the deeper levels of thinking and understanding that a constructivist approach provokes. They were also aware of the need to move their own role from giving information and leading the lesson to facilitator and guide. They wanted to step back from the position of lecturer to one of expert learner.

[1] ***Teaching and Learning Approaches*** is a key document supporting the new curriculum, available from the Framework and in *Secondary mathematics guidance papers* distributed at the summer 2008 SLDM.

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The agreed outcomes for this work were to:

- improve pupils' thinking and reasoning skills;
- improve pupils' understanding of enlargement from a given point;
- improve teachers' skills in using 'cognitive conflict' to promote learning.

Implementation

The teachers first observed a demonstration lesson on patterns and sequences, which highlighted the misconception that all sequences are linear. They noted how the pupils responded to this as they struggled to construct new meaning. Their discussions with the LA consultant produced the following reflections which became guiding principles in their planning:

- in order to challenge thinking the teacher needs to be aware of the understanding that already exists, therefore planning should provide opportunities for assessment of prior learning;
- it is useful to anticipate the misconceptions that might arise;
- a misconception will surface when prior understanding does not fit new learning;
- the classroom culture needs to be one in which experimentation and 'thinking out loud' is encouraged;
- by providing structured examples, the pupils can be led to construct a hypothesis that will fit for the examples encountered so far but is not a true generalisation of the bigger picture;

> How do you deal with moments of cognitive conflict? What are effective probing questions to ask?

- cognitive conflict will only arise if the pupil has the opportunity to test their hypothesis and consequently find that it is inadequate. The structure of the lesson must, therefore, provide some examples that do not fit the hypothesis. This is the moment when cognitive conflict is provoked: the teacher's role is to lead the pupil to adjust their original hypothesis to fit, so constructing new meaning.

The teachers then sought to apply these observations and reflections to their own planning of lessons, setting up situations which would cause pupils to question their own understanding of established concepts.

The first lesson planned for their Year 8 groups focused on enlargement. The misconception anticipated was that an enlargement of factor 2 would always double all the coordinates of a shape drawn on a coordinate grid. Changing the centre of enlargement from its position at the origin would provoke cognitive conflict and pupils would need to search to find other generalisations to fit an enlargement from any centre.

> Does your departmental development plan include opportunities for peer observation and mutual reflection?

During the lesson the pupils were required to work collaboratively on a sequence of tasks, and to engage in mathematical talk. The teachers observed each other's lessons focusing on pupils' reactions to moments of cognitive conflict.

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Impact on learners

Initially most pupils tended to think quietly to themselves, preferring to resolve the conflict alone. However, having worked through the process they then became very excited and at this stage desperately wanted to share their ideas with their peers: this provided the motivation for them to work and discuss together in small groups.

> How would you group pupils for this stage of the activity?

Pupils seemed more willing to work at finding a resolution to this type of challenge than if a difficult question had been posed by the teacher. The teachers conjectured that this might be because the challenge is self-imposed and not external. The conflict arises because of some 'internal error of thinking', therefore resolution becomes more of a personal challenge.

The surprise element of successfully resolving the conflict and creating new meaning caused great excitement and this was seen as both motivating and memorable for the pupils.

> Can you think of any well known misconceptions that might benefit from this sort of approach?

Impact on teachers

The process of unpicking the moments leading up to 'cognitive conflict' resulted in much deep thinking by the teachers. It refined their planning process significantly, not only to meet desired learning outcomes but also to ensure that thinking takes place and to define more precisely when and how this would happen.

Next steps

The two teachers are continuing the experimental process with other topics and classes in order to embed constructivist learning further into their everyday practice. They are involving others in the department in discussion on the areas of mathematics where misconceptions commonly occur.

> How do you react to misunderstandings and misconceptions without damaging pupils' confidence?
>
> Do you discuss this with others in your department?

Further references

If you would like to know more about different teaching models or ideas for activities designed to raise the thinking power of pupils you may like to read:

- *Pedagogy and Practice*, Unit 2: Teaching models, *Secondary mathematics planning toolkit,* 'Pedagogy' folder
- *Thinking Maths, Cognitive Acceleration in Maths Education* (2007), Adhami, M. and Shayer, M. Available from Heinemann, ISBN 978-0-435307-80-6.

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Older and wiser

This study describes how a mathematics department in a large mixed comprehensive school contributed to a whole-school initiative to encourage independent research. The department was also working towards more fully addressing the new curriculum by expanding pupils' awareness of the cultural and historical roots of mathematics. The pedagogical approach chosen was 'teaching to develop enquiry skills'.

What it did for us …

Pupil	***"I liked being able to find things out for myself. I was interested to find out how some of my mathematics was discovered."***
Teacher	***"They were really engaged by being allowed to research for themselves and deciding on their own presentations. I'm going to use this strategy more often, especially when introducing a new topic area."***

Why read this?

This is an example of how to develop those principles of *Teaching and Learning Approaches*[1] outlined below:

Pupils:

- learn about and learn through the key mathematical processes;
- work collaboratively and engage in mathematical talk.

Teachers:

- use rich collaborative tasks;
- use technology in appropriate ways;
- expose pupils to the cultural and historical roots of mathematics.

Context

Pupils were asked to research a historical mathematician, and to present their findings in ways that would interest their peers.

The agreed outcomes for this work were to:

- help pupils to appreciate the importance of the cultural and historical roots of mathematics;
- develop pupils' skills in using mathematical vocabulary to present their research and explain their findings;
- help teachers to consider their role in developing pupils' enquiry skills.

[1] ***Teaching and Learning Approaches*** is a key document supporting the new curriculum, available from the Framework and in *Secondary mathematics guidance papers* distributed at the summer 2008 SLDM.

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Implementation

What are the benefits to learning of pupils researching for themselves? What are the possible pitfalls? What are the cross-subject links that could help here?

Pupils were asked to research the contribution made to mathematics by a famous mathematician and to present their findings to their peers in any way they chose. Pupils were given suggestions of research subjects such as Pythagoras, Hypatia, Newton and Archimedes, but could also make independent choices.

Pupils worked collaboratively in groups, but produced individual presentations. They used the internet to research, but were strongly discouraged from cutting and pasting. Class time was used to discuss how to present their findings in an interesting and attractive way, and the presentations were completed in homework time over a two-week period.

They were free to choose how to represent the information they had found. Some chose to create a poster, some wrote poems/rhymes; others used technology in a variety of ways.

In this instance the presentations were judged by teachers. What else could have been added by involving the pupils in judging and establishing criteria for judging?

The project was given a high profile within the school. A prize was awarded for the best presentation (voted by the staff) which was presented at Prize Day. The pupils' work was displayed publicly in the school foyer as well as in the mathematics area. Interesting facts were published in the school magazine along with winners' names.

Initially the project was intended to run on a monthly basis, known as 'Mathematician of the Month'. However, this proved difficult to sustain, so the task was restricted to a shorter project completed during the annual Mathematics Week. This has now run for two consecutive years.

Impact on learners

What other topics in the mathematics curriculum could be used to develop pupils' enquiry skills?

Pupils were very motivated by the challenge. They enjoyed deciding their own lines of enquiry and relating their current learning to an historical context. They were genuinely fascinated by the way that famous mathematicians had arrived at their theorems, and keen to find out whether mathematicians today worked in the same way. The open-ended, learner-centred nature of the task enabled pupils to explore the topic to a higher level than might be achieved in a more structured, task-based enquiry.

The high profile of the project around the school meant that the pupils could see that their efforts were recognised and realised that they were working to inform a wider audience. This made them work hard on their use of mathematical vocabulary and symbolic language within their presentations.

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Involvement in this project has helped underachieving pupils to gain access to mathematics and experience success. The winning presentation came from a pupil previously described by her teacher as 'somewhat mathematically demotivated'.

Impact on teachers

Teachers were pleased and surprised by the high level of pupil engagement. They saw the positive impact on learning of guiding pupils to develop their enquiry and presentation skills, rather than always structuring the work for them. They used some of the pupils' findings to liven up their lessons and built in references to them in the scheme of work.

Next steps

The teachers reported the need to encourage a greater variety of presentation in future and the use of strategies to avoid over-copying from the internet.

They also noted that the learning benefits would be greater if the presentations were completed using cooperative small-group work in class time. They discussed how they could develop pupils' critical evaluation skills using self- and peer-assessment.

What strategies could you use to avoid internet copying? Which other subject departments could offer guidance on this?

What is the role of the teacher in helping pupils to develop the skills of positive critical evaluation? Which other subject teachers could collaborate with the mathematics department to develop these skills?

Further references

If you would like to know more about different teaching models, you may like to read:

- *Pedagogy and Practice*, Unit 2: Teaching models, *Secondary mathematics planning toolkit,* 'Pedagogy' folder.

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Thinking about the gap?

A group of teachers from five schools worked together on a project which aimed explicitly to 'close the gap' between English and mathematics attainment at GCSE. The project aimed to raise attainment for a group of Year 11 pupils whose tracking data indicated they were on the C/D borderline for mathematics and who were therefore likely to miss the chance of attaining five or more A* to C grades at GCSE, including English and mathematics.

The group aimed to use a pedagogical approach of teaching for deductive thinking to help pupils transfer their success from other subjects into their mathematics learning. They decided to use thinking skills strategies and collaborative, cross-subject approaches drawn from the Leading in Learning (LiL) materials with the aim of improving these pupils' thinking skills in mathematics.

What it did for us …

Pupil	***"I think I'm much better at sharing my ideas now. Everyone listens to each other's ideas, even the teacher!"***
Teacher	***"I am really pleased with how much more actively these pupils engage. They are much more prepared to volunteer for tasks, to develop their own methods, to listen and question."***
Consultant	***"It was rewarding to discover how quickly all the teachers in the project valued teaching the LiL lessons. It motivated them to review their teaching styles, and helped them to assess pupil learning rather than just coverage."***

Why read this?

This is an example of how to develop those principles of *Teaching and Learning Approaches*[1] outlined below:

Pupils:

- learn about and learn through the key mathematical processes;
- work on sequences of tasks;
- select the mathematics to use;
- work collaboratively and engage in mathematical talk.

Teachers:

- use cooperative small group work;
- use rich collaborative tasks;
- create connections between mathematical topics.

[1] ***Teaching and Learning Approaches*** is a key document supporting the new curriculum, available from the Framework and in *Secondary mathematics guidance papers* distributed at the summer 2008 SLDM.

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Context

The project 'Raising attainment in thinking skills' (RATTS), involved teachers from five mathematics departments, coordinated by an LA consultant. All the departments had been identified as needing to raise attainment in mathematics at Key Stage 4. The working group of teachers set out to use LiL approaches with targeted Key Stage 4 pupils.

The agreed outcomes for this work were to:

- improve pupil progress from Key Stage 3 to Key Stage 4 with an increase in the proportion of pupils achieving five or more A* to C grades at GCSE, including English and mathematics;
- improve pupils' engagement with mathematics through involvement in a greater variety of learning experiences;
- enable pupils to transfer thinking skills to different subjects and contexts;
- give teachers a better understanding of the pedagogical approach of 'teaching for deductive thinking'.

Implementation

At the outset of the project, the working group discussed the LiL thinking skills approach to planning lessons. They agreed to:

- identify a common thinking skill to develop across mathematics and at least one other subject;
- plan pairs or trios of lessons which would be delivered across the different subjects using a common teaching strategy to address the agreed thinking skill and link the subjects;
- focus on pupils' awareness of the thinking skill and transferability across the curriculum and to other aspects of their lives.

At the initial launch session the teachers agreed to follow the LiL model: identifying a target group of learners across mathematics and two other subject areas. Support from the schools' senior management teams was critical in ensuring that there would be suitable arrangements for joint planning, observations and review across the subject areas.

The teachers subsequently met in half-termly workshops to develop their skills in teaching the LiL strategies so that pupils' thinking was developing. Each time they focused on a new aspect of the thinking skill and planned with teachers from another subject area.

The mathematics teachers planned to strengthen information processing by using the strategies of 'reading images' and 'living graphs'. The intention was to enable the teachers to assess pupil progression in one thinking skill and to work across the substrands in information processing.

Examples of both strategies are available in the *Leading in Learning mathematics subject exemplification*, on the *Secondary mathematics planning toolkit disk*, 'Rich tasks' folder. Two examples are shown below.

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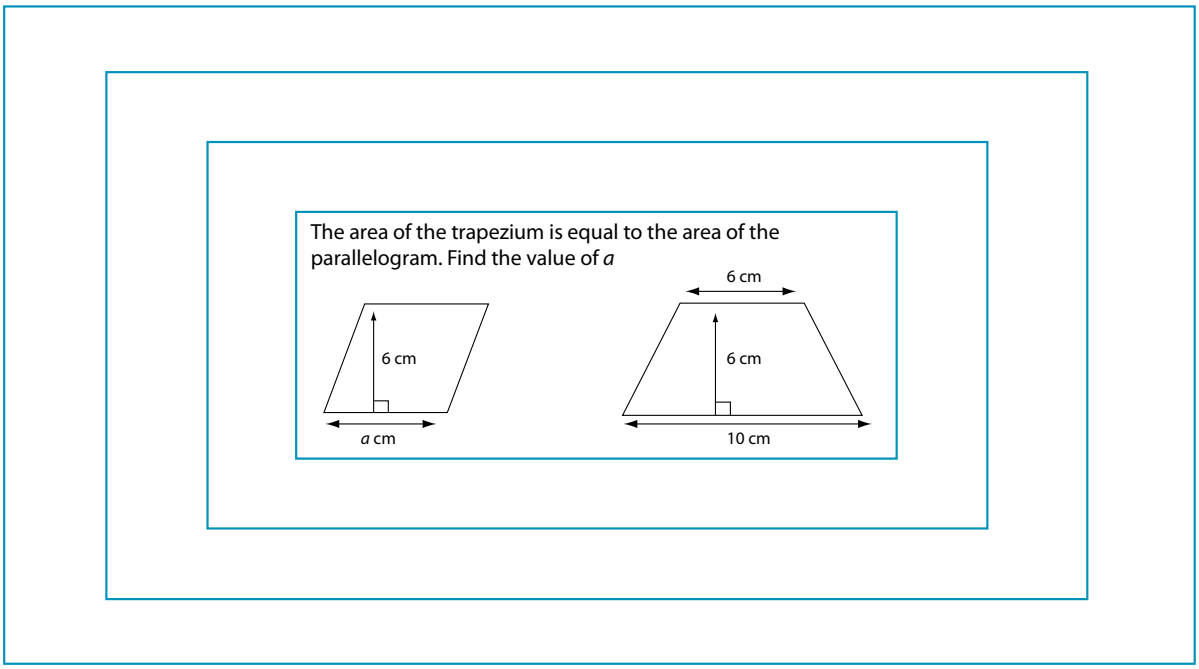
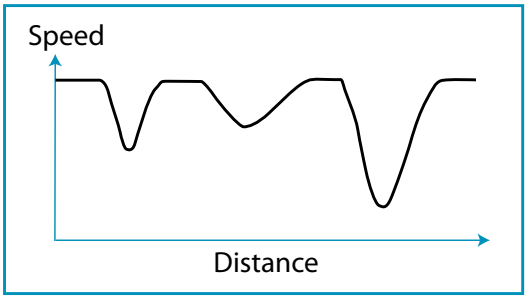
Figure 1 Reading images



A mathematical image is used to stimulate interpretation by groups of pupils. This diagram was used as an activity with Year 10 pupils and was shared as part of SLDM in autumn 2007. 'Reading images' can be used to draw together prior learning or to produce a structure for new learning. Pupils are asked to interpret the image in different stages in each border and finally to give the image a title.

Figure 2 Living graph



'Living graphs' is an activity in which prepared images prompt pupils to think around a 'real-life' context. It leads to pupils composing their own interpretations usually by placing cards along an unscaled graph, and ultimately constructing the graph for themselves. This diagram is an example from *Teaching mental mathematics from level 5 algebra*, available in the *Secondary mathematics planning toolkit*, 'Rich tasks' folder.

The teachers used relevant images appropriate to their scheme of work. Pupils were required to work collaboratively, to make decisions and to peer assess. The teachers actively encouraged originality and invention.

Do you provide opportunities for pupils to reflect on their thinking processes? (metacognition)

In each episode of the unit, the teachers explicitly discussed with the pupils the thinking skills employed and helped them to see links between their thinking in the different activities and in other subject areas. This is described as 'bridging' in the LiL materials.

The deductive model involves pupils testing initial hypotheses by analysing the data to confirm or refute. How does the bridging activity in a LiL lesson support deductive thinking?

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Impact on learners

How do the different LiL strategies promote pupils' process skills in mathematics?

The targeted Year 11 pupils will sit their GCSEs in summer 2008 so, at the time of writing, GCSE results are yet to be confirmed. However, teachers noted an improvement in pupils' metacognitive responses through informal classroom assessment. The pupils were significantly better able to express their thinking with appropriate language and to bridge effectively across their learning. Teachers across the subjects involved in the schools observed improved skills in elements of information processing such as locating and collecting relevant information, sorting and classifying, sequencing, comparing and contrasting and analysing part/whole relationships.

Impact on teachers

The teachers felt that the experience has given them skills that will help them to plan effectively to develop pupils' functional skills in mathematics and to track progress in conjunction with colleagues in other departments. The perceived benefits to pupils' learning have led to a greater willingness to engage in cross-curricular collaborative work.

Next steps

The project schools now intend to use the lessons learned to help them turn their attention to improving learning in Key Stage 3. They want to develop key processes across different subject areas and build towards greater collaboration in teaching and learning functional skills.

Further references

If you would like to know more about different teaching techniques and sources of ideas, you may like to read:

- *Pedagogy and Practice*, Unit 2: Teaching models, *Secondary mathematics planning toolkit,* 'Pedagogy' folder
- *Leading in Learning*; *Developing thinking skills in secondary schools*, *Secondary mathematics planning toolkit*, 'Rich tasks' folder
- *Teaching mental mathematics from level 5 algebra*, *Secondary mathematics planning toolkit,* 'Rich tasks' folder
- *Teaching and learning functional mathematics: Resources to support the pilot of functional skills*, *Secondary mathematics planning toolkit*, 'Rich tasks' folder.

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Wellington and London, Greece and Ireland

This study was developed by a small group of consultants using a familiar Key Stage 3 resource to illustrate aspects of the new programme of study. The activity provides opportunities for pupils to develop their process skills in statistics by considering the advantages and disadvantages of different graphical representations and critically evaluating information presented graphically. The pedagogical approach used is 'teaching to develop enquiry skills'.

What it did for us …

Consultant ***"I realised that many of the existing resources could be used to help subject leaders develop schemes of work which address the new programme of study."***

Consultant ***"It stopped me from looking for a 'magic answer' to support schools with the new curriculum. We need to follow through with the good resources we already have."***

Why read this?

This is an example of how to develop those principles of *Teaching and Learning Approaches* [1] outlined below:

Pupils:

- learn about and learn through the key mathematical processes;
- select the mathematics to use;
- work collaboratively and engage in mathematical talk.

Teachers:

- build on the knowledge pupils bring to a sequence of lessons;
- expose and discuss misconceptions;
- use technology in appropriate ways.

[1] ***Teaching and Learning Approaches*** is a key document supporting the new curriculum, available from the Framework and in *Secondary mathematics guidance papers* distributed at the summer 2008 SLDM.

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Context

The task is informed by two graphs without labels or titles. Pupils are asked to create and then iteratively refine hypotheses about the graphs through a series of prompts as more information is gradually revealed. 'Wellington and London' shows a bar chart and 'Greece and Ireland' shows a pair of pie charts. The resource is described in *Interacting with mathematics in Key Stage 3: Handling data* (available on the *Secondary mathematics planning toolkit disk*, in the 'Rich tasks' folder).

> Do you use graphs from other curricular areas, for example, geography or science, to enrich mathematical learning?

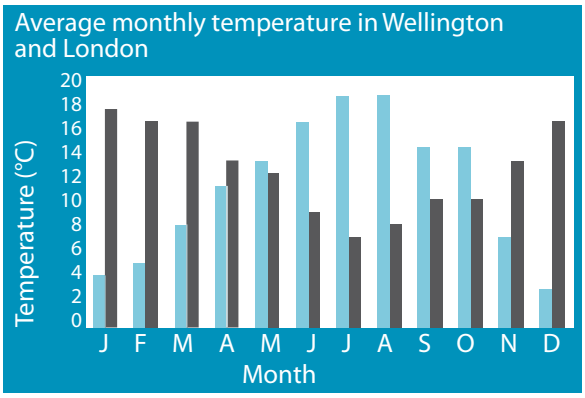
The tasks set around the graphs can empower pupils to understand the advantages and disadvantages of a graphical representation and expose potential misconceptions in interpreting charts.

The outcomes that have been observed from this activity include:

- pupils using their creativity to hypothesise as they assign meanings to different graphical representations;
- pupils developing an understanding of the situations described by the graphs as they use their skills of interpretation;
- pupils justifying their reasoning as they explore possible misconceptions;
- teachers focusing on enabling pupils to develop analysing and reasoning skills.

Implementation

Figure 1 'Wellington and London'



> Do you always tell pupils what a graph is about or do you invite them to form hypotheses?

Pupils are initially shown the above graph but without axes scales, labels or a graph title. They are invited to discuss the potential situations which the graph could represent. They work in pairs or small groups to suggest various hypotheses which can then be tested. As the prompts on the slides are revealed, they are encouraged to re-evaluate their hypotheses in the light of the new information. The teacher emphasises that at any stage there are a variety of valid hypotheses which are progressively eliminated as new information is revealed.

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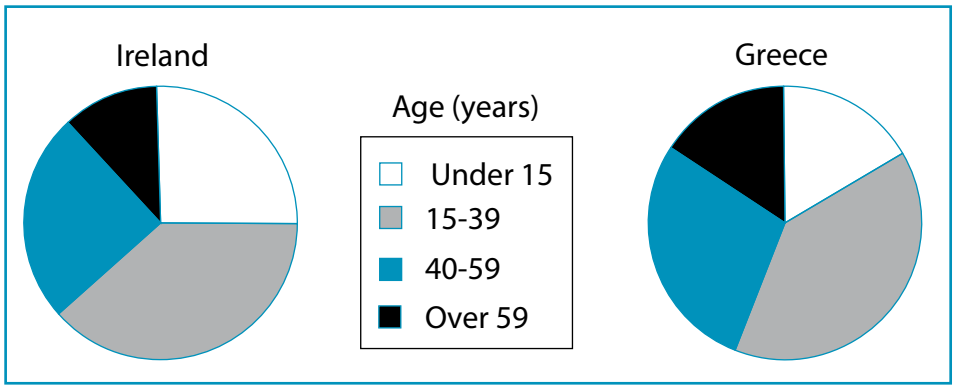
Figure 2 'Greece and Ireland'



The slide above issues a further challenge as pupils are asked:

Do the charts show that:

a) there are more people aged under 15 in Ireland than in Greece

b) there are fewer people aged under 15 in Greece than in Ireland

c) there is a higher proportion of people aged under 15 in Ireland than in Greece?

This allows pupils to expose and discuss the common misconception that sectors on pie charts can provide comparison of absolute totals rather than comparative proportions.

> What is the teacher's role in this episode of the lesson?

Impact on learners

The enquiry-based approach gives pupils the opportunity to hypothesise as they construct possible meanings. In this way:

- their analysis skills are developed;
- they feel they have control over the mathematics;
- they are more able to interpret existing graphs after assigning meaning to partially-labelled graphs;
- they use higher-order thinking skills to process and sort the information.

> Can you think of other topics where an enquiry approach would enhance pupil learning?

Impact on teachers

Teachers focus more on enabling pupils to develop analysing and reasoning skills than on the teaching of decontextualised statistical procedures.

> Think about how you would use 'mini-plenaries' throughout this type of activity to develop reasoning skills.

Next steps

The experience of discussing the interpretations for these graphs makes pupils more skilled in selecting appropriate graphical representation for their own data. It may now be appropriate to:

- give pupils some situations to display graphically;
- ask pupils to identify potentially misleading graphical representations in the media;
- give pupils opportunities to formulate questions to ask about graphs and charts;
- use the teaching strategies 'living graphs' and 'reading images' to further develop thinking around visual images. Both are available in the Leading in Learning (LiL) handbook.

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Further references

If you would like to know more about a range of different teaching models or teaching strategies to develop thinking skills, you may like to read:

- *Pedagogy and Practice*, Unit 2: Teaching models, *Secondary mathematics planning toolkit,* 'Pedagogy' folder
- *Interacting with mathematics in Key Stage 3: handling data, Secondary mathematics planning toolkit,* 'Rich tasks' folder
- *Leading in Learning; Developing thinking skills in secondary schools*, *Secondary mathematics planning toolkit,* 'Rich tasks' folder.

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Appendix

Relational understanding and instrumental understanding *

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Faux amis

Faux amis is a term used by the French to describe words which are the same, or very alike, in two languages, but whose meanings are different. For example:

French word	*Meaning in English*
histoire	story, not history
libraire	bookshop, not library
chef	head of any organisation, not only chief cook
agrément	pleasure or amusement, not agreement
docteur	doctor (higher degree), not medical practitioner
médecin	medical practitioner, not medicine
parent	relations in general, including parents

One gets *faux amis* between English as spoken in different parts of the world. An Englishman asking in America for a biscuit would be given what we call a scone. To get what we call a biscuit, he would have to ask for a cookie. And between English as used in mathematics and in everyday life there are such words as field, group, ring, ideal.

A person who is unaware that the word he is using is a *faux ami* can make inconvenient mistakes. We expect history to be true, but not a story. We take books without paying from a library, but not from a bookshop; and so on. But in the foregoing examples there are cues which might put one on guard: difference of language, or of country, or of context.

If, however, the same word is used in the same language, country and context, with two meanings whose difference is non-trivial but as basic as the difference between the meaning of (say) 'histoire' and 'story', which is a difference between fact and fiction, one may expect serious confusion. Two such words can be identified in the context of mathematics; and it is the alternative meanings attached to these words, each by a large following, which in my belief are at the root of many of the difficulties in mathematics education today.

One of these is 'understanding'. It was brought to my attention some years ago by Stieg Mellin-Olsen, of Bergen University, that there are in current use two meanings of this word. These he distinguishes by calling them 'relational understanding' and 'instrumental understanding'. By the former is meant what I have always meant by understanding, and probably most readers of this article: knowing both what to do and why. Instrumental understanding I would until recently not have regarded as understanding at all. It is what I have in the past described as 'rules without reasons', without realising that for many

*First published in *Mathematics Teaching*, (1976) **77**, 20-26.

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pupils *and their teachers* the possession of such a rule, and ability to use it, was what they meant by 'understanding'.

Suppose that a teacher reminds a class that the area of a rectangle is given by $A = L \times B$. A pupil who has been away says he does not understand, so the teacher gives him an explanation along these lines. "The formula tells you that to get the area of a rectangle, you multiply the length by the breadth." "Oh, I see," says the child, and gets on with the exercise. If we were now to say to him (in effect) "You may think you understand, but you don't really," he would not agree. "Of course I do. Look; I've got all these answers right." Nor would he be pleased at our devaluing of his achievement. And with his meaning of the word, he does understand.

We can all think of examples of this kind: 'borrowing' in subtraction, 'turn it upside down and multiply' for division by a fraction, 'take it over to the other side and change the sign', are obvious ones; but once the concept has been formed, other examples of instrumental explanations can be identified in abundance in many widely used texts. Here are two from a text used by a former direct-grant grammar school, now independent, with a high academic standard.

Multiplication of fractions To multiply a fraction by a fraction, multiply the two numerators together to make the numerator of the product, and the two denominators to makes its denominator.

E.g $\frac{2}{3}$ of $\frac{4}{5} = \frac{2\times4}{3\times5} = \frac{8}{15}$

$\frac{3}{5} \times \frac{10}{13} = \frac{30}{65} = \frac{6}{13}$

The multiplication sign x is generally used instead of the word 'of '.

Circles The circumference of a circle (that is its perimeter, or the length of its boundary) is found by measurement to be a little more than three times the length of its diameter. In any circle the circumference is approximately 3.1416 times the diameter, which is roughly $3\frac{1}{7}$ times the diameter. Neither of these figures is exact, as the exact number cannot be expressed either as a fraction or a decimal. The number is represented by the Greek letter π.

Circumference = πd or $2\pi r$

Area = πr^2

The reader is urged to try for himself this exercise of looking for and identifying examples of instrumental explanations, both in texts and in the classroom. This will have three benefits. (i) For persons like the writer, and most readers of this article, it may be hard to realise how widespread is the instrumental approach. (ii) It will help, by repeated examples, to consolidate the two contrasting concepts. (iii) It is a good preparation for trying to formulate the difference in general terms. Result (i) is necessary for what follows in the rest of the present section, while (ii) and (iii) will be useful for the others.

If it is accepted that these two categories are both well-filled, by those pupils and teachers whose goals are respectively relational and instrumental understanding (by the pupil), two questions arise. First, does this matter? And second, is one kind better than the other? For years I have taken for granted the answers to both these questions: briefly, 'Yes; relational.' But the existence of a large body of experienced teachers and of a large number of texts belonging to the opposite camp has forced me to think more about why I hold this view. In the process of changing the judgement from an intuitive to a reflective one, I think I have learnt something useful. The two questions are not entirely separate, but in the present section I shall concentrate as far as possible on the first: does it matter?

The problem here is that of a mis-match, which arises automatically in any *faux ami* situation, and does not depend on whether A or B's meaning is 'the right one'. Let us imagine, if we can, that school A send a team to play school B at a game called 'football', but that neither team knows that there are two kinds (called 'association' and 'rugby'). School A plays soccer and has never heard of rugger, and vice versa for B. Each team will rapidly decide that the others are crazy, or a lot of foul players. Team A in particular will think that B uses a mis-shapen ball, and commit one foul after another. Unless the two sides stop

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and talk about what game they think they are playing at, long enough to gain some mutual understanding, the game will break up in disorder and the two teams will never want to meet again.

Though it may be hard to imagine such a situation arising on the football field, this is not a far-fetched analogy for what goes on in many mathematics lessons, even now. There is this important difference, that one side at least cannot refuse to play. The encounter is compulsory, on five days a week, for about 36 weeks a year, over ten years or more of a child's life.

Leaving aside for the moment whether one kind is better than the other, there are two kinds of mathematical mis-matches which can occur.

1. Pupils whose goal is to understand instrumentally, taught by a teacher who wants them to understand relationally.
2. The other way about.

The first of these will cause fewer problems short-term to the pupils, though it will be frustrating to the teacher. The pupils just won't want to know all the careful groundwork he gives in preparation for whatever is to be learnt next, nor his careful explanations. All they want is some kind of rule for getting the answer. As soon as this is reached, they latch on to it and ignore the rest.

If the teacher asks a question that does not quite fit the rule, of course they will get it wrong. For the following example I have to thank Mr. Peter Burney, at that time a student at Coventry College of Education on teaching practice. While teaching 'area', he became suspicious that the children did not really understand what they were doing. So he asked them: "What is the area of a field 20 cms by 15 yards?" The reply was: "300 square centimetres." He asked: "Why not 300 square yards?" Answer: "Because area is always in square centimetres."

To prevent errors like the above the pupils need another rule (or, of course, relational understanding), that both dimensions must be in the same unit. This anticipates one of the arguments which I shall use against instrumental understanding, that it usually involves a multiplicity of rules rather than fewer principles of more general application.

There is of course always the chance that a few of the pupils will catch on to what the teacher is trying to do. If only for the sake of these, I think he should go on trying. By many, probably a majority, his attempts to convince them that being able to use the rule is not enough will not be well received. 'Well is the enemy of better,' and if pupils can get the right answers by the kind of thinking they are used to, they will not take kindly to suggestions that they should try for something beyond this.

The other mis-match, in which pupils are trying to understand relationally but the teaching makes this impossible, can be a more damaging one. An instance which stays in my memory is that of a neighbour's child, then seven years old. He was a very bright little boy, with an I.Q. of 140. At the age of five he could read *The Times*, but at seven he regularly cried over his mathematics homework. His misfortune was that he was trying to understand relationally teaching which could not be understood in this way. My evidence for this belief is that when I taught him relationally myself, with the help of Unifix, he caught on quickly and with real pleasure.

A less obvious mis-match is that which may occur between teacher and text. Suppose that we have a teacher whose conception of understanding is instrumental, who for one reason or other is using a text which aim is relational understanding by the pupil. It will take more than this to change his teaching style. I was in a school which was using my own text[1], and noticed (they were at Chapter 1 of *Book* 1) that some of the pupils were writing answers like 'the set of {flowers}'.

When I mentioned this to the teacher (he was head of mathematics), he asked the class to pay attention to him and said: "Some of you are not writing your answers properly. Look at the example in the book, at the beginning of the exercise, and be sure you write you answers exactly like that."

Much of what is being taught under the description of "modern mathematics" is being taught and learnt just as instrumentally as were the syllabi which have been replaced. This is predictable from the difficulty of accommodating (restructuring) our existing schemas[2]. To the extent that this is so, the innovations have probably done more harm than good, by introducing a mis-match between the

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teacher and the aims implicit in the new content. For the purpose of introducing ideas such as sets, mappings and variables is the help which, rightly used, they can give to relational understanding. If pupils are still being taught instrumentally, then a 'traditional' syllabus will probably benefit them more. They will at least acquire proficiency in a number of mathematical techniques which will be of use to them in other subjects, and whose lack has recently been the subject of complaints by teachers of science, employers and others.

Near the beginning I said that two *faux amis* could be identified in the context of mathematics. The second one is even more serious; it is the word 'mathematics' itself. For we are not talking about better and worse teaching of the same kind of mathematics. It is easy to think this, just as our imaginary soccer players who did not know that their opponents were playing a different game might think that the other side picked up the ball and ran with it because they could not kick properly, especially with such a mis-shapen ball. In which case they might kindly offer them a better ball and some lessons on dribbling.

It has taken me some time to realise that this is not the case. I used to think that maths teachers were all teaching the same subject, some doing it better than others.

I now believe that *there are two effectively different subjects being taught under the same name, 'mathematics'*. If this is true, then this difference matters beyond any of the differences in syllabi which are so widely debated. So I would like to try to emphasise the point with the help of another analogy.

Imagine that two groups of children are taught music as a pencil-and-paper subject. They are all shown the five-line stave, with the curly 'treble' sign at the beginning; and taught that marks on the lines are called E, G, B, D, F. Marks between the lines are called F, A, C, E. They learn that a line with an open oval is called a minim, and is worth two with blacked-in ovals which are called crotchets, or four with blacked-in ovals and a tail which are called quavers, and so on – musical multiplication tables if you like. For one group of children, all their learning is of this kind and nothing beyond. If they have a music lesson a day, five days a week in school terms, and are told that it is important, these children could in time probably learn to write out the marks for simple melodies such as God Save the Queen and Auld Lang Syne, and to solve simple problems such as 'What time is this in?' and 'What key?', and even 'Transpose this melody from C major to A major.' They would find it boring, and the rules to be memorised would be so numerous that problems like 'Write a simple accompaniment for this melody' would be too difficult for most. They would give up the subject as soon as possible, and remember it with dislike.

The other group is taught to associate certain sounds with these marks on paper. For the first few years these are audible sounds, which they make themselves on simple instruments. After a time they can still imagine the sounds whenever they see or write the marks on paper. Associated with every sequence of marks is a melody, and with every vertical set a harmony. The keys C major and A major have an audible relationship, and a similar relationship can be found between certain other pairs of keys. And so on. Much less memory work is involved, and what has to be remembered is largely in the form of related wholes (such as melodies) which their minds easily retain. Exercises such as were mentioned earlier ('Write a simple accompaniment') would be within the ability of most. These children would also find their learning intrinsically pleasurable, and many would continue it voluntarily, even after O-level or C.S.E.

For the present purpose I have invented two non-existent kinds of 'music lesson', both pencil-and-paper exercises (in the second case, after the first year or two). But the difference between these imaginary activities is no greater than that between two activities which actually go on under the name of mathematics. (We can make the analogy closer, if we imagine that the first group of children were initially taught sounds for the notes in a rather half-hearted way, but that the associations were too ill-formed and unorganised to last.) The above analogy is, clearly, heavily biased in favour of relational mathematics. This reflects my own viewpoint. To call it a viewpoint, however, implies that I no longer regard it as a self-evident truth which requires no justification: which it can hardly be if many experienced teachers continue to teach instrumental mathematics. The next step is to try to argue the merits of both points of view as clearly and fairly as possible; and especially those of the point of view opposite to one's own. This is why the next section is called Devil's Advocate. In one way this only describes that part which puts the case for instrumental understanding. But it also justifies the other

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part, since an imaginary opponent who thinks differently from oneself is a good device for making clearer to oneself why one does think this way.

Devil's Advocate

Given that so many teachers teach instrumental mathematics, might this be because it does have certain advantages? I have been able to think of three advantages (as distinct from situational reasons for teaching this way, which will be discussed later).

1. Within its own context, instrumental mathematics is usually easier to understand; sometimes much easier. Some topics, such as multiplying two negative numbers together, or dividing by a fractional number, are difficult to understand relationally. "Minus times minus equals plus" and "to divide by a fraction you turn it upside down and multiply" are easily remembered rules. If what is wanted is a page of right answers, instrumental mathematics can provide this more quickly and easily.
2. So the rewards are more immediate, and more apparent. It is nice to get a page of right answers, and we must not underrate the importance of the feeling of success which pupils get from this. Recently I visited a school where some of the children describe themselves as 'thickos'. Their teachers use the term too. These children need success to restore their self-confidence, and it can be argued that they can achieve this more quickly and easily in instrumental mathematics than in relational.
3. Just because less knowledge is involved, one can often get the right answer more quickly and reliably by instrumental thinking than relational. This difference is so marked that even relational mathematicians often use instrumental thinking. This is a point of much theoretical interest, which I hope to discuss more fully on a future occasion.

The above may well not do full justice to instrumental mathematics. I shall be glad to know of any further advantages which it may have. There are four advantages (at least) in relational mathematics.

4. *It is more adaptable to new tasks*. Recently I was trying to help a boy who had learnt to multiply two decimal fractions together by dropping the decimal point, multiplying as for whole numbers, and re-inserting the decimal point to give the same total number of digits after the decimal point as there were before. This is a handy method if you know why it works. Through no fault of his own, this child did not; and not unreasonably, applied it also to division of decimals. By this method $4.8 \div 0.6$ came to 0.08. The same pupil had also learnt that if you know two angles of a triangle, you can find the third by adding the two given angles together and subtracting from 180°. He got ten questions right this way (his teacher believed in plenty of practise), and went on to use the same method for finding the exterior angles. So he got the next five answers wrong.

   I do not think he was being stupid in either of these cases. He was simply extrapolating from what he already knew. But relational understanding, by knowing not only what method worked but why, would have enabled him to relate the method to the problem, and possibly to adapt the method to new problems. Instrumental understanding necessitates memorising which problems a method works for and which not, and also learning a different method for each new class of problems. So the first advantage of relational mathematics leads to:

5. *It is easier to remember*. There is a seeming paradox here, in that it is certainly harder to learn. It is certainly easier for pupils to learn that 'area of a triangle = $\frac{1}{2}$ base × height' than to learn why this is so. But they then have to learn separate rules for triangles, rectangles, parallelograms, trapeziums; whereas relational understanding consists partly in seeing all these in relation to the area of a rectangle. It is still desirable to know the separate rules; one does not want to have to derive them afresh every time. But knowing also how they are inter-related enables one to remember them as parts of a connected whole, which is easier.

   There is more to learn – the connections as well as the separate rules – but the result, once learnt, is more lasting. So there is less re-learning to do, and in the long-term the time taken may well be less altogether.

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Teaching for relational understanding may also involve more actual content. Earlier, an instrumental explanation was quoted leading to the statement 'Circumference = πd'. For relational understanding of this, the idea of a proportion would have to be taught first (among others), and this would make it a much longer job than simply teaching the rules as given. But proportionality has such a wide range of other applications that it is worth teaching on these grounds also. In relational mathematics this happens rather often. Ideas required for understanding a particular topic turn out to be basic for understanding many other topics too. Sets, mappings and equivalence are such ideas.

Unfortunately the benefits which might come from teaching them are often lost by teaching them as separate topics, rather than as fundamental concepts by which whole areas of mathematics can be interrelated.

6. *Relational knowledge can be effective as a goal in itself.* This is an empiric fact, based on evidence from controlled experiments using non-mathematical material. The need for external rewards and punishments is greatly reduced, making what is often called the 'motivational' side of the teacher's job much easier. This is related to:
7. *Relational schemas are organic in quality.* This is the best way I have been able to formulate a quality by which they seem to act as an agent of their own growth. The connection with advantage 3 is that if people get satisfaction from relational understanding, they may not only try to understand relationally new material which is put before them, but also actively seek out new material and explore new areas, very much like a tree extending its roots or an animal exploring new territory in search of nourishment. To develop this idea beyond the level of an analogy is beyond the scope of the present paper, but it is too important to leave out. If the above is anything like a fair presentation of the cases for the two sides, it would appear that while a case might exist for instrumental mathematics short-term and within a limited context, long-term and in the context of a child's whole education it does not. So why are so many children taught only instrumental mathematics throughout their school careers? Unless we can answer this, there is little hope of improving the situation.

An individual teacher might make a reasoned choice to teach for instrumental understanding on one or more of the following grounds.

1. That relational understanding would take too long to achieve, and to be able to use a particular technique is all that these pupils are likely to need.
2. That relational understanding of a particular topic is too difficult, but the pupils still need it for examination reasons.
3. That a skill is needed for use in another subject (e.g. science) before it can be understood relationally with the schemas presently available to the pupil.
4. That he is a junior teacher in a school where all the other mathematics teaching is instrumental.

All of these imply, as does the phrase 'make a reasoned choice', that he is able to consider the alternative goals of instrumental and relational understanding on their merits and in relation to a particular situation.

To make an informed choice of this kind implies awareness of the distinction, and relational understanding of the mathematics itself. So nothing else but relational understanding can ever be adequate for a teacher. One has to face the fact that this is absent in many who teach mathematics; perhaps even a majority.

Situational factors which contribute to the difficulty include:

1. *The backwash effect of examinations*. In view of the importance of examinations for future employment, one can hardly blame pupils if success in these is one of their major aims. The way pupils work cannot but be influenced by the goal for which they are working, which is to answer correctly a sufficient number of questions.

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2. *Over-burdened syllabi*. Part of the trouble here is the high concentration of the information content of mathematics. A mathematical statement may condense into a single line as much as in another subject might take over one or two paragraphs. By mathematicians accustomed to handling such concentrated ideas, this is often overlooked (which may be why most mathematics lecturers go too fast). Non-mathematicians do not realise it at all. Whatever the reason, almost all syllabi would be much better if much reduced in amount so that there would be time to teach them better.
3. *Difficulty of assessment* of whether a person understands relationally or instrumentally. From the marks he makes on paper, it is very hard to make valid inference about the mental processes by which a pupil has been led to make them; hence the difficulty of sound examining in mathematics. In a teaching situation, talking with the pupil is almost certainly the best way to find out; but in a class of over 30, it may be difficult to find the time.
4. *The great psychological difficulty for teachers of accommodating (re-structuring) their existing and long-standing schemas*, even for the minority who know they need to, want to do so, and have time for study.

From a recent article discussing the practical, intellectual and cultural value of a mathematics education (and I have no doubt that he means relational mathematics!) by Sir Hermann Bondi[3], I take these three paragraphs. (In the original, they are not consecutive.)

> So far my glowing tribute to mathematics has left out a vital point: the rejection of mathematics by so many, a rejection that in not a few cases turns to abject fright.
>
> The negative attitude to mathematics, unhappily so common, even among otherwise highly-educated people, is surely the greatest measure for our failure and a real danger to our society.
>
> This is perhaps the clearest indication that something is wrong, and indeed very wrong, with the situation. It is not hard to blame education for at least a share of the responsibility; it is harder to pinpoint the blame, and even more difficult to suggest new remedies.

If for 'blame' we may substitute 'cause', there can be small doubt that the widespread failure to teach relational mathematics – a failure to be found in primary, secondary and further education, and in 'modern' as well as 'traditional' courses – can be identified as a major cause. To suggest new remedies is indeed difficult, but it may be hoped that diagnosis is one good step towards a cure. Another step will be offered in the next section.

A Theoretical Formulation

There is nothing so powerful for directing one's actions in a complex situation, and for coordinating one's own efforts with those of others, as a good theory. All good teachers build up their own stores of empirical knowledge, and have abstracted from these some general principles on which they rely for guidance. But while their knowledge remains in this form it is largely still at the intuitive level within individuals, and cannot be communicated, both for this reason and because there is no shared conceptual structure (schema) in terms of which it can be formulated. Were this possible, individual efforts could be integrated into a unified body of knowledge which would be available for use by newcomers to the profession. At present most teachers have to learn from their own mistakes.

For some time my own comprehension of the difference between the two kinds of learning which lead respectively to relational and instrumental mathematics remained at the intuitive level, though I was personally convinced that the difference was one of great importance, and this view was shared by most of those with whom I discussed it. Awareness of the need for an explicit formulation was forced on me in the course of two parallel research projects; and insight came, quite suddenly, during a recent conference. Once seen it appears quite simple, and one wonders why I did not think of it before. But there are two kinds of simplicity: that of naivety; and that which, by penetrating beyond superficial differences, brings simplicity by unifying. It is the second kind which a good theory has to offer, and this is harder to achieve.

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A concrete example is necessary to begin with. When I went to stay in a certain town for the first time, I quickly learnt several particular routes. I learnt to get between where I was staying and the office of the colleague with whom I was working; between where I was staying and the university refectory where I ate; between my friend's office and the refectory; and two or three others. In brief, I learnt a limited number of fixed plans by which I could get from particular starting locations to particular goal locations. As soon as I had some free time, I began to explore the town. Now I was not wanting to get anywhere specific, but to learn my way around, and in the process to see what I might come upon that was of interest. At this stage my goal was a different one: to construct in my mind a cognitive map of the town.

These two activities are quite different. Nevertheless they are, to an outside observer, difficult to distinguish. Anyone seeing me walk from A to B would have great difficulty in knowing (without asking me) which of the two I was engaged in. But the most important thing about an activity is its goal. In one case my goal was to get to B, which is a physical location. In the other it was to enlarge or consolidate my mental map of the town, which is a state of knowledge. A person with a set of fixed plans can find his way from a certain set of starting points to a certain set of goals. The characteristic of a plan is that it tells him what to do at each choice point: turn right out of the door, go straight on past the church, and so on. But if at any stage he makes a mistake, he will be lost; and he will stay lost if he is not able to retrace his steps and get back on the right path.

In contrast, a person with a mental map of the town has something from which he can produce, when needed, an almost infinite number of plans by which he can guide his steps from any starting point to any finishing point, provided only that both can be imagined on his mental map. And if he does take a wrong turn, he will still know where he is, and thereby be able to correct his mistake without getting lost; even perhaps to learn from it.

The analogy between the foregoing and the learning of mathematics is close. The kind of learning which leads to instrumental mathematics consists of the learning of an increasing number of fixed plans, by which pupils can find their way from particular starting points (the data) to required finishing points (the answers to the questions). The plan tells them what to do at each choice point, as in the concrete example. And as in the concrete example, what has to be done next is determined purely by the local situation. (When you see the post office, turn left. When you have cleared brackets, collect like terms.) There is no awareness of the overall relationship between successive stages, and the final goal. And in both cases, the learner is dependent on outside guidance for learning each new 'way to get there'. In contrast, learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point. (I say 'in principle' because of course some of these paths will be much harder to construct than others.)

This kind of learning is different in several ways from instrumental learning.

1. The means become independent of particular ends to be reached thereby.
2. Building up a schema within a given area of knowledge becomes an intrinsically satisfying goal in itself.
3. The more complete a pupil's schema, the greater his feeling of confidence in his own ability to find new ways of 'getting there' without outside help.
4. But a schema is never complete. As our schemas enlarge, so our awareness of possibilities is thereby enlarged. Thus the process often becomes self-continuing, and (by virtue of 3) self-rewarding.

Taking again for a moment the role of devil's advocate, it is fair to ask whether we are indeed talking about two subjects, relational mathematics and instrumental mathematics, or just two ways of thinking about the same subject matter. Using the concrete analogy, the two processes described might be regarded as two different ways of knowing about the same town; in which case the distinction made between relational and instrumental understanding would be valid, but not between instrumental and relational mathematics.

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But what constitutes mathematics is not the subject matter, but a particular kind of knowledge about it. The subject matter of relational and instrumental mathematics may be the same: cars travelling at uniform speeds between two towns, towers whose heights are to be found, bodies falling freely under gravity, etc etc. But the two kinds of knowledge are so different that I think that there is a strong case for regarding them as different kinds of mathematics. If this distinction is accepted, then the word 'mathematics' is for many children indeed a false friend, as they find to their cost.

The State of Play

This is already a long article, yet it leaves many points awaiting further development. The applications of the theoretical formulation in the last section to the educational problems described in the first two have not been spelt out. One of these is the relationship between the goals of the teacher and those of the pupil. Another is the implications for a mathematical curriculum.

In the course of discussion of these ideas with teachers and lecturers in mathematical education, a number of other interesting points have been raised which also cannot be explored further here. One of these is whether the term 'mathematics' ought not to be used for relational mathematics only. I have much sympathy with this view, but the issue is not as simple as it may appear.

There is also research in progress. A pilot study aimed at developing a method (or methods) for evaluating the quality of children's mathematical thinking has been finished, and has led to a more substantial study in collaboration with the N.F.E.R. as part of the TAMS continuation project. A higher degree thesis at Warwick University is nearly finished; and a research group of the Department of Mathematics at the University of Quebec in Montreal is investigating the problem with first and fourth grade children. All this will I hope be reported in due course.

The aims of the present paper are twofold. First, to make explicit the problem at an empiric level of thinking, and thereby to bring to the forefront of attention what some of us have known for a long time at the back of our minds. Second, to formulate this in such a way that it can be related to existing theoretical knowledge about the mathematical learning process, and further investigated at this level and with the power and generality which theory alone can provide.

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2. For a fuller discussion see R. R. Skemp: *The Psychology of Learning Mathematics* (Penguin 1972) pp. 43–46.
3. H. Bondi: *The Dangers of Rejecting Mathematics (Times Higher Education Supplement*, 26.3.76).

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